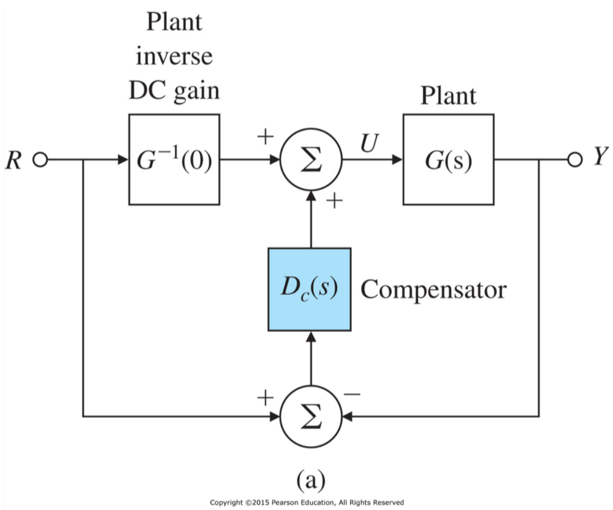
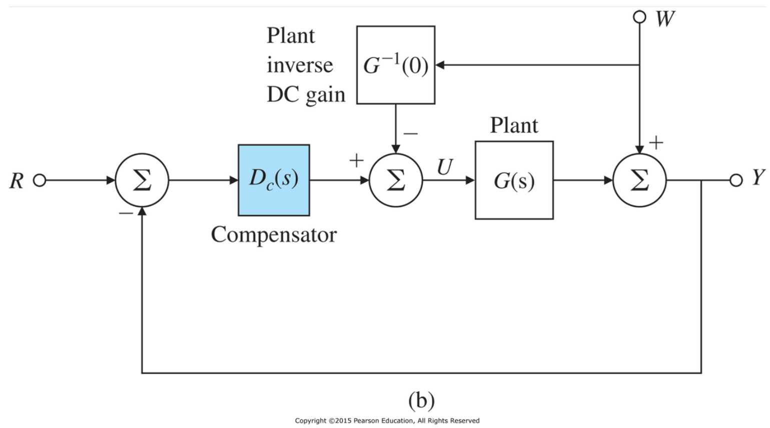
- (Question) fig 4.27(b)에서 전달함수 $\frac{Y}{W}$ 식을 구하는 과정 ?



(a)

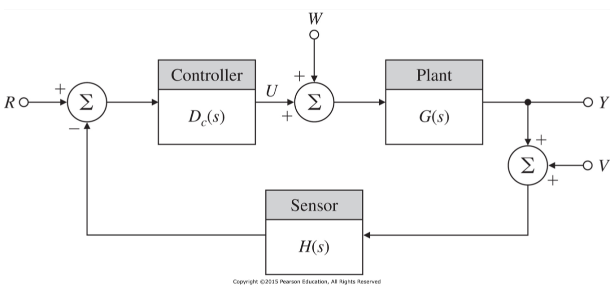


(b)

제 5 장

The Root-Locus Design Method

1 Root Locus of a Basic Feedback System



- The characteristic equation can be rearranged with the parameter of interest K:

$$1 + D_c(s)G(s)H(s) = 0 \quad \rightarrow \quad a(s) + Kb(s) = 0 \quad \rightarrow \quad 1 + KL(s) = 0 \text{ with } L(s) = \frac{b(s)}{a(s)} \quad \rightarrow \quad L(s) = -\frac{1}{K}$$

  where it is noted that K can be the gain of the controller.

  - The locus of all possible roots of the characteristic equation is plotted as K varies from zero to infinity, and then we can use the resulting plot to aid us in selecting the best value of K in viewpoints of stability and performance.
  - The solutions of above equations are the roots (poles) of the closed-loop system.

- Let us factor the monic polynomials $a(s)$ and $b(s)$ as

$$a(s) = s^n + a_1 s^{n-1} + \cdots + a_n = (s - p_1)(s - p_2) \cdots (s - p_n)$$
$$b(s) = s^m + b_1 s^{m-1} + \cdots + b_m = (s - z_1)(s - z_2) \cdots (s - z_m)$$

where p_i and z_i are pole and zero of $L(s)$, not the pole and zero of the closed-loop system. The roots of the characteristic equation itself are r_i from the factored form ($n > m$)

$$a(s) + Kb(s) = (s - r_1)(s - r_2) \cdots (s - r_n)$$

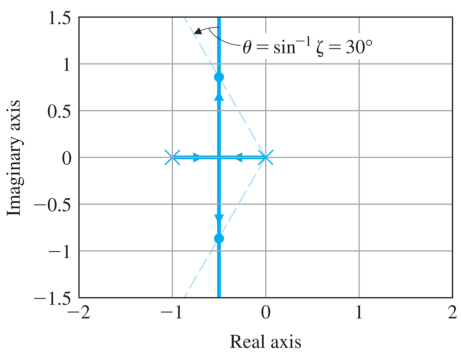
where r_i is pole of the closed-loop system.

- (Example 5.1) In Fig. 5.1, assume that $G(s) = \frac{A}{s(s+1)}$ and $D_c(s) = H(s) = 1$. Root Locus wrt. A?

  1. $L(s) = \frac{1}{s(s+1)}$ and $K = A$
  2. $a(s) = s^2 + s$ with $p_1 = -1$, $p_2 = 0$ and $b(s) = 1$ with no zero
  3. characteristic equation and closed-loop poles:

$$a(s) + Kb(s) = s^2 + s + K = 0 \qquad r_{1,2} = \frac{-1 \pm \sqrt{1-4K}}{2}$$

     - at $K = 0$, the roots are $s = -1$ and $s = 0$.
     - for $0 < K < \frac{1}{4}$, the roots are real between -1 and 0
     - at $K = \frac{1}{4}$, two repeated roots at $s = -\frac{1}{2}$ (breakaway point)
     - for $K > \frac{1}{4}$, the roots become complex with real parts at $-\frac{1}{2}$ and imaginary parts that increase essentially in proportion to the square root of K.



  4. The dashed lines in Fig. 5.2 correspond to roots with a damping ratio $\zeta = 0.5$ ($\theta = \sin^{-1}\zeta = 30°$). The crossing points denoted by dots can be calculated as follows:

$$r_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{4K-1}}{2}j = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}j \qquad \rightarrow \qquad \therefore \quad K = 1$$

- solve (Example 5.2)
- (Example) In the Fig. 5.1, assume that $G(s) = \frac{1}{s(s+c)}$ and $D_c(s) = H(s) = 1$. Root Locus wrt c ?

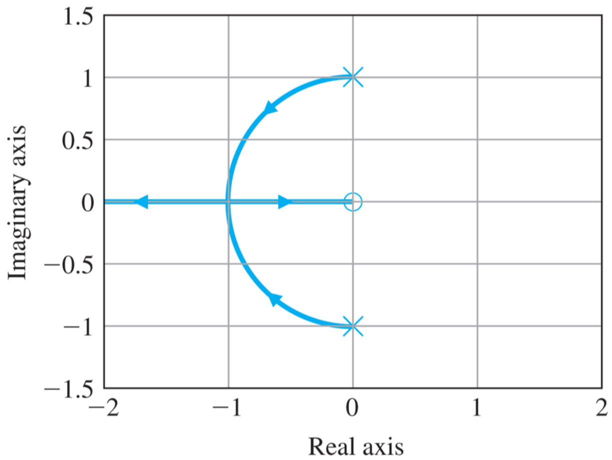
1. The closed-loop characteristic equation:

$$1 + G(s) = 1 + \frac{1}{s^2 + cs} = 0 \qquad \rightarrow \qquad 1 + c\frac{s}{s^2+1} = 0$$

2. $L(s) = \frac{s}{s^2+1}$ and $K = c$
3. $a(s) = s^2 + 1$ with $p_{1,2} = \pm j$ and $b(s) = s$ with $z_1 = 0$
4. characteristic equation and closed-loop poles:

$$a(s) + Kb(s) = s^2 + Ks + 1 = 0 \qquad r_{1,2} = \frac{-K \pm \sqrt{K^2 - 4}}{2}$$

   - at $K = 0$, the roots are $s = j$ and $s = -j$
   - for $0 < K < 2$, the roots are complex at $s = -\frac{K}{2} \pm \frac{\sqrt{4-K^2}}{2}j$.
   - at $K = 2$, two repeated roots at $s = -1$ (break-in point)
   - for $K > 2$, the roots become real values on the negative real axis at $s = -\frac{K}{2} \pm \frac{\sqrt{K^2-4}}{2}$
   - as $K \rightarrow \infty$, the real roots approach at $s = 0$ and $s = -\infty$.



5. For the understanding of locus of the complex roots, let us apply $s = \sigma + j\omega$ for $0 < K < 2$:

$$s^2 + Ks + 1 = \sigma^2 - \omega^2 + 2j\sigma\omega + K(\sigma + j\omega) + 1 = 0 \quad \rightarrow \quad \sigma^2 - \omega^2 + K\sigma + 1 = 0 \text{ and } 2\sigma\omega + K\omega = 0$$

From above relation, we can know $K = -2\sigma$ and we can derive the following:

$$\sigma^2 - \omega^2 + K\sigma + 1 = \sigma^2 - \omega^2 - 2\sigma^2 + 1 = 0 \quad \rightarrow \quad \sigma^2 + \omega^2 = 1 \quad \text{for} \quad -1 < \sigma < 0$$

thus we can know that the semi-circle is plotted for $0 < K < 2$ as shown in the figure.

- Matlab command : $rlocus(sys)$

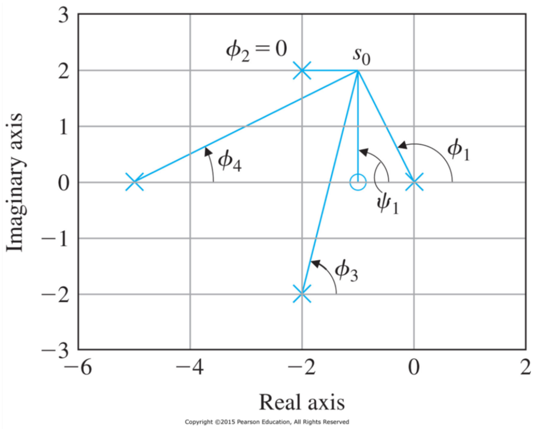
2 Guidelines for Determining a Root Locus

- (Definition I) The root locus is the set of values of s for which $1 + KL(s) = 0$ is satisfied as the real parameter K varies from 0 to $+\infty$. Typically, $1 + KL(s) = 0$ is the characteristic equation of the system, and in this case, the roots on the locus are the closed-loop poles of that system.

- (Definition II, Phase Condition) The root locus of $L(s)$ is the set of points in the s-plane where the phase of $L(s)$ is $180°$. To test whether a point in the s-plane is on the locus, we define the angle to the test point from a zero as ψ_i and the angle to the test point from a pole as ϕ_i then the Definition II is expressed as those points in the s-plane where, for an integer l,

$$\angle L(s_0) = \sum \psi_i - \sum \phi_i = 180° + 360(l - 1)$$

$$\sum \text{angle to the test point from a zero} - \sum \text{angle to the test point from a pole} = \pm 180°, \pm 540°, \cdots$$



- Consider the example,

$$L(s) = \frac{s+1}{s(s+5)[(s+2)^2+4]}$$

In the figure, the poles are marked $\times$ and the zero is marked $\bigcirc$. Suppose we select the test point $s_0 = -1+2j$. Let us test whether or not s_0 (test point) lies on the root locus for some value of K.

$$\psi_1 = \angle(s_0 - (-1)) = \angle((-1+2j)-(-1)) = \angle 2j = 90^\circ$$

$$\phi_1 = \angle(s_0 - (0)) = \angle(-1+2j) = 180^\circ - \tan^{-1} 2 = 116.6^\circ$$

$$\phi_2 = \angle(s_0 - (-2+2j)) = \angle((-1+2j)-(-2+2j)) = \angle 1 = 0^\circ$$

$$\phi_3 = \angle(s_0 - (-2-2j)) = \angle((-1+2j)-(-2-2j)) = \angle(1+4j) = \tan^{-1} 4 = 76^\circ$$

$$\phi_4 = \angle(s_0 - (-5)) = \angle(4+2j) = \tan^{-1}\frac{1}{2} = 26.6^\circ$$

as a result $\quad \angle L = \psi_1 - (\phi_1+\phi_2+\phi_3+\phi_4) = -129.2^\circ \neq -180^\circ$

Since the phase of $L(s_0)$ is not $\pm 180^\circ$, we conclude that s_0 is not on the root locus.

1. Rules for Determining a Positive Root Locus

   a) (Rule 1, Start and End) The n branches of the locus start at the poles of $L(s)$ and m of these branches end on the zeros of $L(s)$.

$$a(s) + Kb(s) = 0$$

$$\text{when } K = 0, \quad a(s) = 0 \quad \text{poles of } L(s) \text{ are roots}$$

$$\text{when } K = \infty, \quad b(s) = 0 \quad \text{zeros (including infinity zeros) of } L(s) \text{ are roots}$$

   b) (Rule 2, Real Axis) The loci are on the real axis to the left of an odd number of poles and zeros.

   c) (Rule 3, Asymptotes) For large s and K, $n - m$ branches of the loci are asymptotic to lines at angles ϕ_l radiating out from the point $s = \alpha$ on the real axis, where

$$\phi_l = \frac{180^\circ + 360(l-1)}{n-m} \quad \text{for} \quad l = 0, \pm 1, \pm 2, \cdots$$

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m}$$

d) (Rule 4, Departure Angles and Arrival Angles) The angle of departure of a branch of the locus from a pole is given by, with the multiplicity q of the repeated poles,

$$
\begin{aligned}
q\phi_{l,dep} &= \sum \psi_i - \sum_{i \neq l,dep} \phi_i - 180^\circ - 360^\circ(l-1) \\
&= \text{sum of the angles to all zeros} - \text{sum of the angles to the remaining poles} - 180^\circ - 360^\circ(l-1)
\end{aligned}
$$

where l is an integer and takes on the values $1, 2, \ldots, q$.
Likewise, the angles of arrival of a branch at a zero with multiplicity q is given by

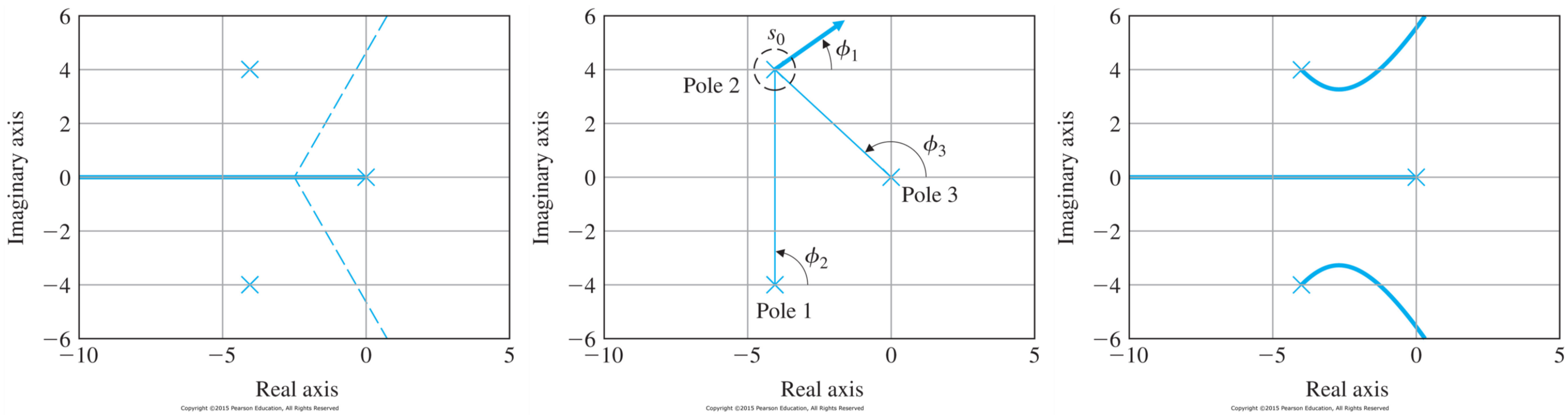
$$
\begin{aligned}
q\psi_{l,arr} &= \sum \phi_i - \sum_{i \neq l,arr} \psi_i + 180^\circ + 360^\circ(l-1) \\
&= \text{sum of the angles to all poles} - \text{sum of the angles to the remaining zeros} + 180^\circ + 360^\circ(l-1)
\end{aligned}
$$

e) (Rule 5, Break-in and Breakaway Points) The break-in and breakaway points are obtained by solving

$$
\frac{dK}{ds} = 0 \qquad \leftarrow \qquad K = -\frac{1}{L(s)} = -\frac{b(s)}{a(s)}
$$

f) Consider the following example:

$$L(s) = \frac{1}{s[(s+4)^2 + 16]}$$



i. (Rule 1, Start and End)

$$\text{when } K = 0, \qquad s = 0, -4+4j, -4-4j \qquad \text{poles of } L(s)$$
$$\text{when } K = \infty, \qquad s = \infty, \infty, \infty \qquad \text{zeros of } L(s)$$

ii. (Rule 2, Real Axis) Negative real axis is locus

iii. (Rule 3, Asymptotes) point at α with angles of ϕ_l (Fig. 5.6)

$$\phi_l = \frac{180^\circ + 360(l-1)}{3} = \pm 60^\circ, 180^\circ$$
$$\alpha = \frac{0 - 4 + 4j - 4 - 4j}{3} = -\frac{8}{3}$$

iv. (Rule 4, Departure Angles and Arrival Angles) (Fig. 5.7)

$$\phi_{dep,-4+4j} = 0 - (\angle(-4+4j-0) + \angle(-4+4j+4+4j)) - 180^\circ = 0 - 135^\circ - 90^\circ - 180^\circ = -45^\circ$$
$$\phi_{dep,-4-4j} = 0 - (\angle(-4-4j-0) + \angle(-4-4j+4-4j)) - 180^\circ = 0 + 135^\circ + 90^\circ - 180^\circ = 45^\circ$$

v. (Rule 5, Break-in and Breakaway Points) No break-away and break-in points.

vi. As a result, the root-locus of the system is given by implementing the following code

```
s = tf('s');
sysL = 1/(s*((s+4)^2+16));
rlocus(sysL)
[K,p] = rlocfind(sysL)
```

2. Selecting the Parameter Value

- Using Definition II of the locus, we have developed rules to sketch a root locus from the phase of $L(s)$ alone. If the equation is actually to have a root at a particular place when the phase of $L(s)$ is $180°$, then a magnitude condition must also be satisfied.
- The magnitude condition is written as

$$K = \frac{1}{|L(s)|} \qquad \leftarrow \qquad K = -\frac{1}{L(s)}$$

- For given the following example, let us calculate the the gain K when $\zeta = 0.5$.

$$L(s) = \frac{1}{s[(s+4)^2+16]}$$

Let us assume that the crossing point $s_0 = -2+2\sqrt{3}j$ between $\zeta = 0.5$ line and the root locus is found as shown in Fig. 5.9. Then,

$$K = \frac{1}{|L(s_0)|} = |s_0| \cdot |s_0 - (-4+4j)| \cdot |s_0 - (-4-4j)| = 4 \cdot 2.1 \cdot 7.7 = 65$$