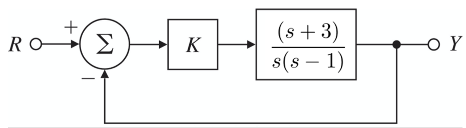
Solutions of Midterm Exam

Subject : Control System Engineering 2, Lecturer : Prof. Youngjin Choi,
Date : Oct. 27, 2020 (Contact e-mail : cyj@hanyang.ac.kr)

Problem 1 (20pt) Determine the stability properties of the following closed-loop system using Nyquist criterion? where it is noted that $K > 0$.



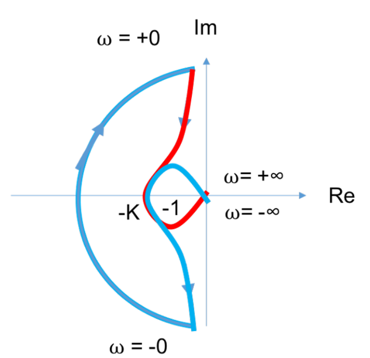
Solution of Problem 1 (20pt) Let us draw Nyquist plot of the following:

$$G(s) = \frac{(s+3)}{s(s-1)} \qquad |KG(j\omega)| = \frac{K\sqrt{\omega^2+9}}{|\omega|\sqrt{\omega^2+1}} \qquad \angle KG(j\omega) = \tan^{-1}\frac{\omega}{3} - 90^\circ - (180^\circ - \tan^{-1}\omega)$$

- $\omega = +0$ 일때, 크기 $= \infty$, 위상각 $= -270^\circ$
- $\omega = +1$ 일때, 크기 $= \sqrt{5}K$, 위상각 $= -206.5^\circ$
- $\omega = +\sqrt{3}$ 일때, 크기 $= K$, 위상각 $= -180^\circ$

$$\tan^{-1}\frac{\omega}{3} + \tan^{-1}\omega = 90^\circ \qquad \frac{\frac{\omega}{3}+\omega}{1-\frac{\omega}{3}\omega} = \infty \qquad \omega = \sqrt{3}$$

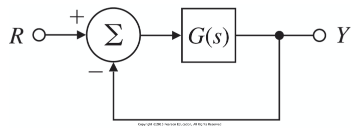
- $\omega = +3$ 일때, 크기 $= \frac{K}{\sqrt{5}}$, 위상각 $= -153.4^\circ$
- $\omega = +\infty$ 일때, 크기 $= 0$, 위상각 $= -90^\circ$
- As $s : +0 \to -0$ with a radius ϵ: $\frac{K(s+3)}{s(s-1)} \approx \frac{3K}{-\epsilon} = -\infty$는 음의 무한대 반원으로 맵핑된다.



Thus, it is

- stable when $K > 1$ because $Z = P + N = 0$ with $P = 1$, $N = -1$
- neutral stable when $K = 1$
- unstable when $0 < K < 1$ because $Z = P + N = 2$ with $P = 1$, $N = 1$

Problem 2 (20pt) Find the phase crossover frequency ω_p, the gain margin GM, the gain crossover frequency ω_g, and the phase margin PM of the following closed-loop system? where $G(s) = \frac{1-s}{s(s+3)}$



Solution of Problem 2 (20pt)

$$G(j\omega) = \frac{1 - j\omega}{j\omega(j\omega + 3)} \qquad |G(j\omega)| = \frac{\sqrt{1+\omega^2}}{|\omega|\sqrt{9+\omega^2}} \qquad \angle G(j\omega) = -\tan^{-1}\omega - 90^\circ - \tan^{-1}\frac{\omega}{3}$$

1. 위상교차 주파수 ω_p

$$\tan^{-1}\omega_p + \tan^{-1}\frac{\omega_p}{3} = 90^\circ \quad \rightarrow \quad \omega_p^2 = 3 \quad \rightarrow \quad \omega_p = \sqrt{3}$$

2. 이득여유 GM

$$|G(j\omega_p)| = \frac{\sqrt{1+\omega_p^2}}{|\omega_p|\sqrt{9+\omega_p^2}} = \frac{1}{3} \qquad GM = \frac{1}{|G(j\omega_p)|} = 3 \qquad GM[dB] = 20\log 3 = 9.54[dB]$$

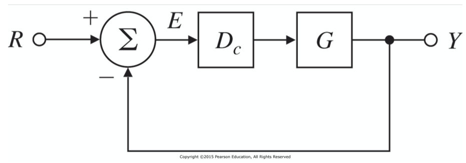
3. 이득교차 주파수 ω_g

$$\frac{\sqrt{1+\omega_g^2}}{|\omega_g|\sqrt{9+\omega_g^2}} = 1 \quad \rightarrow \quad 1+\omega_g^2 = \omega_g^2(9+\omega_g^2) \quad \rightarrow \quad \omega_g^2 = \sqrt{17} - 4 \quad \rightarrow \quad \omega_g = 0.35$$

4. 위상여유 PM

$$PM = 180^\circ - \tan^{-1}\omega_g - 90^\circ - \tan^{-1}\frac{\omega_g}{3} = 64.1^\circ$$

Problem 3 (20pt) For given system $G(s) = \frac{1}{s(s+2)}$, we wish to meet a steady-state error requirement for a unit-ramp input ($K_v = 10$), furthermore, to assure the phase margin of $PM = 40°$. Design the lag compensation $D_c(s) = K\beta\frac{Ts+1}{\beta Ts+1}$ satisfying two specifications? where $\beta > 1$.



Solution of Problem 3 (20pt) $D_c(s) = K\beta\frac{Ts+1}{\beta Ts+1} = K\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$

1. Velocity error constant

$$K_v = \lim_{s\to 0} sD_c(s)G(s) = \lim_{s\to 0} K\beta\frac{Ts+1}{\beta Ts+1}\frac{1}{(s+2)} = \frac{K\beta}{2} = 10 \quad \rightarrow \quad K\beta = 20$$

그러므로

$$G_1(s) = K_c\beta G(s) = \frac{20}{s(s+2)}$$

2. 목표 위상여유가 40도이므로 5도를 더하여 45도를 위상여유로 잡아서 새로운 이득교차주파수를 찾는다.

$$\angle G_1(j\omega_g) = -90° - \tan^{-1}\frac{\omega_g}{2} = -135° \quad \rightarrow \quad \omega_g = 2$$

3. $\omega_g = 2$로 부터 $\frac{1}{T} = \frac{1}{10}\omega_g = 0.2$로 선정. 그러므로 $T = 5$

4. 이득교차주파수에서 크기 = 0 [db]이 되기 위한 β 결정 필요

$$\begin{aligned} 20\log|G_1(j\omega_g)| &= 20\log 10 - 20\log\omega_g - 20\log\sqrt{1+0.25\omega_g^2} \\ &= 20 - 6 - 3 = 11[db] \quad \rightarrow \quad -20\log\beta = -11 \quad \rightarrow \quad \beta = 3.55 \quad \rightarrow \quad K_c = 5.63 \end{aligned}$$

Thus, we have

$$G_c(s) = 5.63\frac{s+0.2}{s+0.056} = 20\frac{5s+1}{17.8s+1}$$

Problem 4 (20pt) Find the state description matrices in the control canonical form and the modal canonical form of the following transfer function, respectively?

$$G(s) = \frac{s+7}{s(s^2+2s+2)}$$

Solution of Problem 4 (20pt)

1. For the control canonical from,

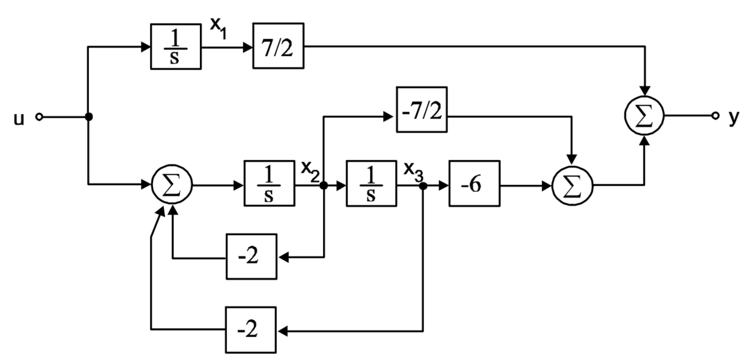
$$G(s) = \frac{0s^2+1s+7}{s^3+2s^2+2s+0}$$

we have

$$\dot{x} = \begin{bmatrix} -2 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \qquad y = \begin{bmatrix} 0 & 1 & 7 \end{bmatrix} x + [0]u$$

2. For the modal canonical from,

$$G(s) = \frac{3.5}{s} + \frac{-3.5s-6}{s^2+2s+2} = G_1(s) + G_2(s) \quad \rightarrow \quad G_1(s) = \frac{3.5}{s} \text{ and } G_2(s) = \frac{-3.5s-6}{s^2+2s+2}$$



we have

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u \qquad y = \begin{bmatrix} 3.5 & -3.5 & -6 \end{bmatrix} x + [0]u$$

Problem 5 (20pt) For given system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]u$$

1. Find the control law that places the closed-loop poles of the system so that they are both at -2 ?

2. Find the output $y(t)$ of the closed-loop control system with initial conditions $x_1(0) = 1$ and $x_2(0) = 0$?

Solution of Problem 5 (20pt)

1. Let us apply the control law

$$u = -\begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then the desired characteristic equation should be equal to $\alpha_c(s) = (s+2)^2$

$$\det[sI - A + BK] = \det\left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix} \right\}$$
$$= s^2 + K_2 s + K_1 + 1 = s^2 + 4s + 4 = \alpha_c(s)$$

By comparing both sides, we have

$$\therefore \quad K_1 = 3 \qquad\qquad K_2 = 4$$

2. Closed-loop control system is obtained as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = (A - BK) \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad\qquad y = x_2$$

If we assume $A_c = A - BK$, the state vector can be found as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = e^{A_c t} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = e^{A_c t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

let us use the matrix exponential

$$e^{A_c t} = \mathcal{L}^{-1}[[sI - A_c]^{-1}] = \mathcal{L}^{-1} \begin{bmatrix} s & -1 \\ 4 & s+4 \end{bmatrix}^{-1} = \mathcal{L}^{-1} \begin{bmatrix} \frac{s+4}{(s+2)^2} & \frac{1}{(s+2)^2} \\ \frac{-4}{(s+2)^2} & \frac{s}{(s+2)^2} \end{bmatrix}$$

we have

$$\therefore \quad y(t) = \mathcal{L}^{-1}\left[\frac{-4}{(s+2)^2} \right] = -4te^{-2t} \quad \text{for } t \geq 0$$