(PID) 6 Performance Tuning

1. Most mechanical systems are described by Lagrangian equation of motion and their controllers consist of the PID. For this purpose, an inverse optimal PID controller can be used:

$$\tau = R^{-1}B^T Px = \left(K + \frac{1}{\gamma^2}I\right)s = \left(K + \frac{1}{\gamma^2}I\right)\left(\dot{e} + K_P e + K_I \int edt\right) \quad \textbf{(137)}$$

   and it brings the *performance limitation* for trajectory tracking system models expressed by

$$M(q)\dot{s} + C(q,\dot{q})s = w(t,x) + u \qquad \dot{x} = A(x,t)x + B(x,t)w + B(t,x)u$$

   where $K > 0, K_P > 0, K_I > 0$, $K_P^2 > 2K_I$ and $u = -\tau$.

2. In the previous lecture, we already obtained *three tuning rules* such as

$$\frac{\sqrt{k}}{\gamma} \propto \max\left(\dot{q}_d\right) \qquad k_P \propto \frac{1}{m}\frac{\sqrt{k}}{\gamma} \qquad k_I \propto \frac{k_P}{m}\frac{\sqrt{k}}{\gamma}$$

3. Further, we will derive *three additional performance tuning rules* of an inverse optimal PID control from its performance limitation.

   - Square tuning rule
   - Linear tuning rule
   - Compound tuning rule

(PID) 6.1 Performance Limitation

1. (Refer to Result of Theorem 5.3) If the PID controller (137) is applied to trajectory tracking mechanical system as proved in Theorem 5.3, then the derivative of the Lyapunov function has the following form:

$$\dot{V} \leq -\frac{1}{2}x^T \left(Q + PBKB^T P\right) x + \gamma^2 |w|^2. \tag{138}$$

2. (Extended Disturbance) The extended disturbance of (110) can be expressed as a function of time and state vector as following form:

$$\begin{aligned} w(x,t) &= M(q)\left(\ddot{q}_d + K_P \dot{e} + K_I e\right) + C(q,\dot{q})\left(\dot{q}_d + K_P e + K_I \int e dt\right) + g(q) + d(t) \\ &= \begin{bmatrix} CK_I & MK_I + CK_P & MK_P \end{bmatrix} \begin{bmatrix} \int e dt \\ e \\ \dot{e} \end{bmatrix} + M\ddot{q}_d + C\dot{q}_d + g + d \\ &= H(x,t)x + h(x,t) \end{aligned} \tag{139}$$

where $H(x,t) \triangleq \begin{bmatrix} CK_I & MK_I + CK_P & MK_P \end{bmatrix}$ and $h(x,t) \triangleq M\ddot{q}_d + C\dot{q}_d + g + d$.

3. (Assumptions for Boundedness) Now, consider the Euclidian norm of extended disturbance of (139). Then we can get some insight such that the extended disturbance can be bounded by the function of Euclidian norm of a state vector under the following two assumptions:

   (A1) : the configuration derivative $\dot{q}$ is bounded

   (A2) : the external disturbance $d(t)$ is bounded.

- by the bounds of $\dot{q}$, the Coriolis and centrifugal matrix $C(q,\dot{q})$ can be bounded, *e.g.*, $|C(q,\dot{q})| \le c_0|\dot{q}|$ with a positive constant c_0.
- we know that the gravitational torque $g(q)$ is bounded if the system stays at the earth, $|g(q)| \le g_0$.
- an inertia matrix $M(q)$ is bounded by its own maximum eigenvalue m, *e.g.*, $|M(q)| \le m$.
- the desired configurations ($q_d, \dot{q}_d, \ddot{q}_d$) are specified as the bounded values, $|q_d| < \infty$, $|\dot{q}_d| < \infty$ and $|\ddot{q}_d| < \infty$.

4. (Norm Bound of Extended Disturbance) Then, we can derive the following relationship from above assumptions:

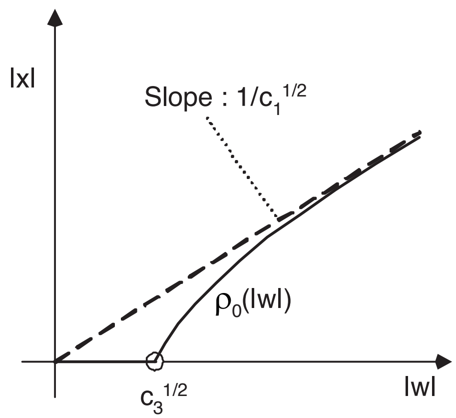
$$\begin{aligned} |w|^2 &= [H(x,t)x + h(x,t)]^T[H(x,t)x + h(x,t)] = x^T(H^TH)x + 2(h^TH)x + (h^Th) \\ &\le c_1|x|^2 + c_2|x| + c_3 \end{aligned} \tag{140}$$

where c_1, c_2 and c_3 are some positive constants.

5. (Shape of Extended Disturbance) The norm of the extended disturbance can be *upper* bounded by the function of that of state vector, conversely, the norm of a state vector can be *lower* bounded by the inverse function of that of extended disturbance:

$$|w| \leq \rho_o^{-1}(|x|) \quad \leftrightarrows \quad \rho_o(|w|) \leq |x|,$$

where $\rho_o(|w|) = 0$ for $0 \leq |w| \leq \sqrt{c_3}$ because, when $0 \leq |w| \leq \sqrt{c_3}$, necessarily $x = 0$.



- Also, the constant c_3 of (140) cannot be zero either in the case of a trajectory tracking control or in the presence of the external disturbances and the gravitational torques.
- Though the function ρ_o must be a continuous, unbounded and increasing function, ρ_o is not a class $\mathcal{K}_\infty$ function because it is not strictly increasing as shown in above figure.
- If there exist no external disturbances ($d(t) = 0$) and the gravity torques ($g(q) = 0$), then the GAS can be proved for the set-point regulation control ($\ddot{q}_d = 0, \dot{q}_d = 0$) because $c_2 = 0, c_3 = 0$ and the function ρ_o becomes a class $\mathcal{K}_\infty$.

6. The control performance is determined by the gain values of a controller, hence, it is important to perceive the relation between the gain values and the errors.

7. (Theorem 6.1) Let $K = kI, K_P = k_P I$ and $K_I = k_I I \in \Re^{n\times n}$. Suppose that λ_{min} is the minimum eigenvalue of following matrix

$$Q_K = Q + PBKB^T P, \tag{141}$$

and that the performance limitation $|x|_{P.L}$ is defined as the Euclidian norm of state vector that satisfies $\dot{V} = 0$. If the PID controller in Theorem 5.3 is applied to the trajectory tracking system model and λ_{min} is chosen sufficiently large and γ sufficiently small so that $\lambda_{min} - 2\gamma^2 c_1 > 0$ can be satisfied, then its performance limitation is upper bounded by

$$|x|_{P.L} \leq \left(\frac{\gamma^2}{\lambda_\gamma}\right)\left[c_2 + \sqrt{c_2^2 + 2c_3\left(\frac{\lambda_\gamma}{\gamma^2}\right)}\right] \tag{142}$$

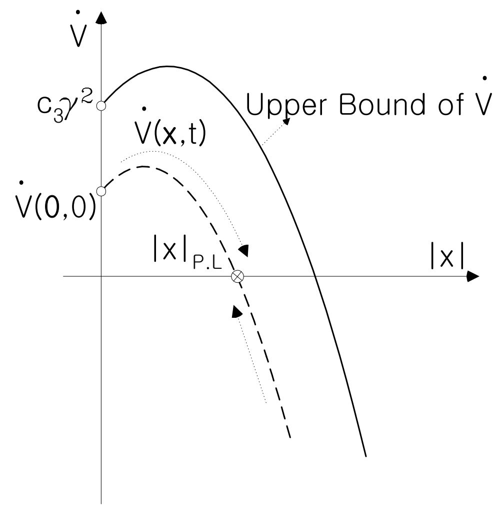
where c_1, c_2, c_3 are coefficients for the *upper* bound of extended disturbance (140), $\lambda_\gamma = \lambda_{min} - 2\gamma^2 c_1$, and the minimum eigenvalue of Q_K is determined by

$$\lambda_{min} \geq \min\left\{k, (k_P^2 - 2k_I)k, k_I^2 k\right\}. \tag{143}$$

This equation (142) can be regarded as the *performance prediction equation* which can predict the performance of the closed-loop system according as the gains change.

8. (Core of proof) From the result of Theorem 5.3,

$$\begin{aligned}\dot{V}(x,t) &\leq -\frac{1}{2}x^T Q_K x + \gamma^2|w|^2 \\ &\leq -\frac{1}{2}\lambda_{min}|x|^2 + \gamma^2(c_1|x|^2 + c_2|x| + c_3) \\ &= -\frac{1}{2}\lambda_\gamma|x|^2 + c_2\gamma^2|x| + c_3\gamma^2. \end{aligned} \quad (144)$$



If we let $\dot{V} = 0$, then we can get

$$|x|_{P.L} \leq \left(\frac{\gamma^2}{\lambda_\gamma}\right)\left[c_2 + \sqrt{c_2^2 + 2c_3\left(\frac{\lambda_\gamma}{\gamma^2}\right)}\right]$$

where $\lambda_\gamma \triangleq \lambda_{min} - 2\gamma^2 c_1 > 0$.

9. Since the *performance limitation* $|x|_{P.L}$ upper bounded by the inequality (142) is the *convergent point* as we can see in above figure, the norm of state vector tends to stay at this point.

10. This analysis can naturally illustrate the performance tuning.

(PID) 6.2 Square and Linear Tuning Rules

1. The exact performance tuning measure is the performance limitation of (142) in Theorem 6.1, however, the coefficients c_1, c_2, c_3 are unknowns.

2. To develop an available tuning rules, let us rewrite the performance limitation (142) using $\sqrt{a^2+b^2} \leq |a| + |b|$:

$$
\begin{aligned}
|x|_{P.L} &\leq \left(\frac{\gamma}{\sqrt{\lambda_\gamma}}\right)^2 \left[c_2 + \sqrt{c_2^2 + 2c_3\left(\frac{\sqrt{\lambda_\gamma}}{\gamma}\right)^2}\right] \\
&\leq \left(\frac{\gamma}{\sqrt{\lambda_\gamma}}\right)^2 \left[2c_2 + \sqrt{2c_3}\left(\frac{\sqrt{\lambda_\gamma}}{\gamma}\right)\right] \\
&= 2c_2\left(\frac{\gamma}{\sqrt{\lambda_\gamma}}\right)^2 + \sqrt{2c_3}\left(\frac{\gamma}{\sqrt{\lambda_\gamma}}\right) \qquad (145)
\end{aligned}
$$

where $\lambda_\gamma = \lambda_{min} - 2\gamma^2 c_1$. For simplicity,

- if γ is chosen sufficiently small, then $\lambda_\gamma \approx \lambda_{min}$.
- if $k_P^2 - 2k_I > 1$ and $k_I > k_P > 1$ can be satisfied, then the λ_{min} is lower bounded by k from (143), *i.e.*, $\lambda_{min} \geq k$.

3. (Tuning Rules) By letting $\lambda_\gamma \approx k$ and defining the *tuning variable* as $\frac{\gamma}{\sqrt{k}}$, the performance limitation of (145) can be expressed by the function of tuning variable $\frac{\gamma}{\sqrt{k}}$ as following form:

$$|x|_{P.L} \propto 2c_2 \left(\frac{\gamma}{\sqrt{k}}\right)^2 + \sqrt{2c_3}\left(\frac{\gamma}{\sqrt{k}}\right). \tag{146}$$

- For a large tuning variable,

$$|x|_{P.L} \propto \gamma^2, \quad \text{for a small } \sqrt{k}. \tag{147}$$

  because the second order term governs the inequality.

- For a small tuning variable,

$$|x|_{P.L} \propto \gamma, \quad \text{for a large } \sqrt{k}, \tag{148}$$

  because the first order term becomes dominant.

- As a summary, we propose two tuning methods;
  - *coarse tuning* that brings the square relation of (147)
  - *fine tuning* that brings the linear relation of (148).

(PID) 6.3 Compound Performance Tuning

1. If the inverse optimal PID controller (137) is applied to the trajectory tracking system (111), then the closed-loop system is obtained as

$$M\dot{s} + Cs = w - \left(K + \frac{1}{\gamma^2}I\right)s$$

2. (Theorem 6.2) Let $s = \dot{e} + K_P e + K_I \int edt \in \Re^n$. If the PID controller (137) is applied to the trajectory tracking system (111), then the composite error is *upper* bounded as following form:

$$|s(t)| \leq |s(0)|e^{-\frac{k\gamma^2+0.5}{\lambda\gamma^2}t} + \frac{\gamma^2}{\sqrt{2k\gamma^2+1}}\|w\|_{\mathcal{L}_\infty}, \qquad (149)$$

where $s(0)$ is the initial composite error vector, λ is a maximum eigenvalue of inertia matrix M, and k is a minimum diagonal element of K.

3. (Proof) Let us differentiate the positive real-valued function

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}s^T M s\right) &= \frac{1}{2}\dot{s}^T M s + \frac{1}{2}s^T \dot{M} s + \frac{1}{2}s^T M \dot{s},\\ &= -s^T\left(K + \frac{1}{\gamma^2}I\right)s + s^T w, \qquad \text{by using } \dot{M} = C + C^T\\ &= -s^T\left(K + \frac{1}{2\gamma^2}I\right)s - \frac{\gamma^2}{2}\left|\frac{1}{\gamma^2}s - w\right|^2 + \frac{\gamma^2}{2}|w|^2,\\ &\le -s^T\left(K + \frac{1}{2\gamma^2}I\right)s + \frac{\gamma^2}{2}|w|^2.\end{aligned}$$

Using the maximum eigenvalue of inertia matrix M and the minimum diagonal element of gain matrix K, above inequality can be simplified to:

$$\frac{d}{dt}\left(\frac{\lambda}{2}|s|^2\right) \le -\left(k + \frac{1}{2\gamma^2}\right)|s|^2 + \frac{\gamma^2}{2}|w|^2.$$

If we multiply above inequality by $e^{\frac{2k\gamma^2+1}{\lambda\gamma^2}t}$, then it becomes

$$\frac{d}{dt}\left(\lambda|s|^2 e^{\frac{2k\gamma^2+1}{\lambda\gamma^2}t}\right) \le \gamma^2|w|^2 e^{\frac{2k\gamma^2+1}{\lambda\gamma^2}t}. \tag{150}$$

Integrating (150) over [0,t], we arrive at the following form:

$$
\begin{aligned}
|s(t)|^2 &\le |s(0)|^2 e^{-\frac{2k\gamma^2+1}{\lambda\gamma^2}t} + \frac{\gamma^2}{\lambda}\int_0^t e^{-\frac{2k\gamma^2+1}{\lambda\gamma^2}(t-\tau)}|w(\tau)|^2 d\tau \\
&\le |s(0)|^2 e^{-\frac{2k\gamma^2+1}{\lambda\gamma^2}t} + \frac{\gamma^2}{\lambda}\sup_{\tau\in[0,t]}\left\{|w(\tau)|^2\right\}\int_0^t e^{-\frac{2k\gamma^2+1}{\lambda\gamma^2}(t-\tau)} d\tau \\
&= |s(0)|^2 e^{-\frac{2k\gamma^2+1}{\lambda\gamma^2}t} + \frac{\gamma^2}{\lambda}\|w\|^2_{\mathcal{L}_\infty}\frac{\lambda\gamma^2}{2k\gamma^2+1}(e^{-\frac{2k\gamma^2+1}{\lambda\gamma^2}t} - 1).
\end{aligned}
$$

By applying the property of $\sqrt{a^2+b^2} \le |a|+|b|$ to right hand side of above inequality, an explicit upper bound of composite error vector is obtained as follow:

$$
\begin{aligned}
|s(t)| &\le |s(0)| e^{-\frac{k\gamma^2+0.5}{\lambda\gamma^2}t} + \frac{\gamma^2}{\sqrt{2k\gamma^2+1}}\|w\|_{\mathcal{L}_\infty}\sqrt{(1 - e^{-\frac{2k\gamma^2+1}{\lambda\gamma^2}t})} \\
&\le |s(0)| e^{-\frac{k\gamma^2+0.5}{\lambda\gamma^2}t} + \frac{\gamma^2}{\sqrt{2k\gamma^2+1}}\|w\|_{\mathcal{L}_\infty} \\
&\le \beta(|s(0)|, t) + \alpha(\|w\|_{\mathcal{L}_\infty})
\end{aligned}
$$

- The first term of right hand side of (149) is a class $\mathcal{KL}$ function $\beta(|s(0)|, t)$ because it is an increasing function for $|s(0)|$ and decreasing one for time t.
- The second term is a class $\mathcal{K}$ function $\alpha(\|w\|_{\mathcal{L}_\infty})$ since it is an increasing one for $\|w\|_{\mathcal{L}_\infty}$.
- Hence, the extended disturbance input-to-state stability (ISS) can be also proved from Theorem 6.2 because the upper bound (149) follows the ISS characteristics.

4. Though the exponential term of (149) goes to zero as $t \to \infty$, the composite error cannot be zero because the extended disturbance (110) includes the inverse dynamics according to desired configurations $(q_d, \dot{q}_d, \ddot{q}_d)$ and gravity force $g(q)$, moreover, $w \neq 0$ as shown in following equation even when $e = 0, \dot{e} = 0, \int edt = 0$:

$$\begin{aligned} w(t, \dot{e}, e, \int edt) =& M\ddot{q}_d + C\dot{q}_d + g + d \\ & + MK_P\dot{e} + (MK_I + CK_P)\, e + CK_I \int edt. \end{aligned} \tag{151}$$

5. As a matter of fact, the upper bound of composite error naturally suggests a new performance tuning rule.

6. In the trajectory tracking control, since the initial composite error $s(0)$ of (149) can be set to zero vector by the initialization of control system, the composite error can be bounded only by $\mathcal{L}_\infty$ norm of the extended disturbance as follows:

$$|s(t)| \leq \frac{\gamma^2}{\sqrt{2k\gamma^2+1}}\|w\|_{\mathcal{L}_\infty}. \tag{152}$$

7. Therefore, if the utilized PID controller can stabilize the system, then we can find the following proportional relation from (152):

$$|s| \propto \frac{\gamma^2}{\sqrt{2k\gamma^2+1}}. \tag{153}$$

   The above is referred to as *compound tuning rule*, and indeed it combines both square and linear tuning rules

8. (Remark 6.2) For a state vector, the square and linear tuning rules were proposed and proved. For a composite error, these square and linear tuning rules can be also found by approximating (153) according to the size of gain k as follows:

$$\begin{aligned} \text{Square Tuning}: \ |s| \propto \gamma^2, &\quad \text{for a small} \quad k, \\ \text{Linear Tuning}: \ |s| \propto \gamma, &\quad \text{for a large} \quad k. \end{aligned}$$

(PID) Summary on Six Gain Tuning Rules of PID Control

When the following PID controller form is utilized for mechanical systems,

$$\tau = \left(k + \frac{1}{\gamma^2}\right) s = \left(k + \frac{1}{\gamma^2}\right)\left(\dot{e} + k_P e + k_I \int e dt\right) \tag{154}$$

we have six performance tuning rules as follows:

1. In Chapter 5, we have three tuning rules

$$\frac{\sqrt{k}}{\gamma} \propto \max(\dot{q}_d) \qquad k_P \propto \frac{1}{m}\frac{\sqrt{k}}{\gamma} \qquad k_I \propto \frac{k_P}{m}\frac{\sqrt{k}}{\gamma}$$

2. In Chapter 6, we have three tuning rules

$$\begin{aligned} |x| &\propto \gamma^2 && \text{for a small } \sqrt{k} \\ |x| &\propto \gamma && \text{for a large } \sqrt{k} \\ |s| &\propto \frac{\gamma^2}{\sqrt{2k\gamma^2+1}}. \end{aligned}$$

   where

$$s = \dot{e} + k_P e + k_I \int e dt \quad \text{and} \quad x = \begin{bmatrix} \int e dt \\ e \\ \dot{e} \end{bmatrix}$$

3. (MatLab Code)

```
close all
clear all

s_time = 0.002;  tf = 1;
q = 0;  qdot = 0;  eint = 0;
qf = 90*(pi/180);  q0 = 0;

global m;
global l;
global g;
global u;
global kf;

m = 1;  l = 1;  g = 9.806;  kf = 0.5;
n=1;

hold on
axis([-1.5 1.5 -1.5 1.5]);
grid
x = l*sin(q);  Ax = [0, x];  y = -l*cos(q);  Ay = [0, y];
p = line(Ax,Ay,'EraseMode','xor','LineWidth',[5],'Color','b');

for t = 0 : s_time : tf

    %%%% Trajectory Generation %%%%
```

```
q_d = q0 + 3*(qf-q0)*t*t - 2*(qf-q0)*t*t*t;
qdot_d = 6*(qf-q0)*t - 6*(qf-q0)*t*t;

%%%% PID after determining the initial gains using ZN method
Kp = 41.83;  Ti = 0.36; Td = 0.09;
%u = Kp*(e + Td*edot + 1/Ti*eint);   % Typical PID form
%u = (K+1/gamma/gamma)*(edot + K_P*e + K_I*eint);   % Inverse Optimal PID
K = Kp*Td;
K_P = 1/Td;
K_I = 1/(Ti*Td);
gamma = 1;

e = q_d-q;
edot = qdot_d - qdot;
eint = eint + e*s_time;
u = (K+1/gamma/gamma)*(edot + K_P*e + K_I*eint);

[tt,z] = ode45('pendulum', [0, s_time], [q; qdot]);
index = size(z); q = z(index(1), 1); qdot = z(index(1), 2);
x = l*sin(q); Ax = [0, x]; y = -l*cos(q); Ay = [0, y];
n=n+1;
data(n+1,1) = t;  data(n+1,2) = q_d; data(n+1,3) = q; data(n+1,4) = q_d-q;

if rem(n,10) == 0
    set(p,'X', Ax, 'Y',Ay)
    drawnow
```

```
    end
end

close all
figure
ax1 = subplot(4,1,1); % top subplot
ax2 = subplot(4,1,2); % mid subplot
ax3 = subplot(4,1,3); % bottom subplot
ax4 = subplot(4,1,4); % bottom subplot

plot(ax1,data(:,1),data(:,2))
ylabel(ax1,'desired');
plot(ax2,data(:,1),data(:,3))
ylabel(ax2,'actual');
plot(ax3,data(:,1),data(:,2),data(:,1),data(:,3))
ylabel(ax3,'desired and actual');
plot(ax4,data(:,1),data(:,4))
ylabel(ax4,'error');

max(data(:,4))
```

- (HW # 10) solve problem 6.1