(PID) 3.2 Completion of Squares

1. For given linear system and performance index to be optimized:

$$\dot{x} = Ax + Bu \qquad J = \lim_{t\to\infty}\left[V(x) + \frac{1}{2}\int_0^t (x^TQx + u^TRu)\, d\tau\right] \quad \text{with } V(x) = \frac{1}{2}x^TPx \tag{78}$$

with $R = R^T > 0$ and $Q = Q^T \geq 0$, using the Riccati equation (77) of $Q = -A^TP - PA + PBR^{-1}B^TP$, the performance index can be manipulated as follows:

$$\begin{aligned}
J &= \lim_{t\to\infty}\left[V(x(t)) + \frac{1}{2}\int_0^t (x^TQx + u^TRu)\, d\tau\right] \\
&= \lim_{t\to\infty}\left[V(x(t)) + \int_0^t \frac{1}{2}x^T(-A^TP - PA + PBR^{-1}B^TP)x + \frac{1}{2}u^TRu\, d\tau\right] \\
&= \lim_{t\to\infty}\left[V(x(t)) - \int_0^t \left(\frac{1}{2}x^T(A^TP + PA)x + x^TPBu\right) d\tau + \frac{1}{2}\int_0^t \left(x^TPBR^{-1}B^TPx + 2x^TPBu + u^TRu\right) d\tau\right] \\
&= \lim_{t\to\infty}\left[V(x(t)) - \int_0^t \dot{V}\, d\tau + \frac{1}{2}\int_0^t \left|R^{\frac{1}{2}}u + R^{-\frac{1}{2}}B^TPx\right|^2 d\tau\right] \\
&= \lim_{t\to\infty}\left[V(x(t)) - V(x(t)) + V(x(0)) + \frac{1}{2}\int_0^t \left|R^{\frac{1}{2}}u + R^{-\frac{1}{2}}B^TPx\right|^2 d\tau\right] \\
&= V(x(0)) + \frac{1}{2}\int_0^\infty \left|R^{\frac{1}{2}}u + R^{-\frac{1}{2}}B^TPx\right|^2 d\tau
\end{aligned}$$

If we choose $u = -R^{-1}B^TPx$, then we get the minimal performance index value

$$\therefore \quad J = V(x(0)) = \text{constant} \quad \Leftarrow \quad u = -R^{-1}B^TPx \tag{79}$$

(PID) 3.3 HJB Equation (Dynamic Programming)

1. (HJB) For given system and performance index to be optimized:

$$\dot{x} = f(x, u, t) \qquad J = \lim_{t\to\infty} \left[V(x(t), t) + \int_0^t g(x(\tau), u(\tau), \tau) d\tau \right] \tag{80}$$

   if the Hamiltonian quantity is defined using the generalized sweep method $\lambda = V_x^T = \frac{\partial V}{\partial x}$ (notice that V_x is a row vector and in the case of linear system $\lambda = Px$ when $V(x) = \frac{1}{2}x^T Px$)

$$H(x, u, V_x, t) = g(x, u, t) + V_x(x, t) f(x, u, t) \tag{81}$$

   then Hamilton-Jacobi-Bellman (HJB) equation is obtained

$$\therefore \quad V_t + \min_u H(x, u, V_x, t) = 0 \qquad \Leftarrow \qquad \frac{\partial H}{\partial u} = 0 \text{ and } \frac{\partial^2 H}{\partial u^2} > 0 \tag{82}$$

   and the nonlinear optimization problem is resolved only if the HJB can be solved.

2. Let us consider the performance index as following form:

$$J = \lim_{t\to\infty} [V(x, t) + \int_0^t g(x, u, \tau) d\tau]$$

   Since $g(x, u, t) > 0$ for all $t \geq 0$, the minimum cost is equal to the Lyapunov function

$$J^*(x, t) \to V(x, t)$$

   In other words, if we can find the Lyapunov function satisfying (82), then we can also find the minimum cost because the minimum cost is equal to the Lyapunov function.

3. (Dynamic Programming for Derivation of HJB)

- For any time interval $t \leq \tau \leq t_f$, the cost (performance index value) is

$$J(x(t), t, u(\tau)_{t\leq\tau\leq t_f}) = V(x(t_f), t_f) + \int_t^{t_f} g(x(\tau), u(\tau), \tau)d\tau$$

- If the minimizing control input is applied in the given time interval, the minimum cost is achieved as follows:

$$\begin{aligned}
J^*(x(t), t) &= \min_{u(\tau)} \left[V(x(t_f), t_f) + \int_t^{t_f} g(x(\tau), u(\tau), \tau)d\tau \right] \\
&= \min_{u(\tau)} \left[V(x(t_f), t_f) + \int_t^{t+\Delta t} g(x(\tau), u(\tau), \tau)d\tau + \int_{t+\Delta t}^{t_f} g(x(\tau), u(\tau), \tau)d\tau \right] \\
&= \min_{u(\tau)} \left[\int_t^{t+\Delta t} g(x(\tau), u(\tau), \tau)d\tau \right] + \min_{u(\tau)} \left[V(x(t_f), t_f) + \int_{t+\Delta t}^{t_f} g(x(\tau), u(\tau), \tau)d\tau \right] \\
&= \min_{u(\tau)} \left[\int_t^{t+\Delta t} g(x(\tau), u(\tau), \tau)d\tau \right] + J^*(x(t+\Delta t), t+\Delta t)
\end{aligned}$$

- If the Taylor series expansion is applied to $J^*(x(t+\Delta t), t+\Delta t)$ at the point $J^*(x(t), t)$, then we have

$$\begin{aligned}
J^*(x(t+\Delta t), t+\Delta t) &= J^*(x(t), t) + J_x^*(x(t), t)[x(t+\Delta t) - x(t)] + J_t^*(x(t), t)\Delta t + \text{higher order} \\
&= J^*(x(t), t) + J_x^*(x(t), t)\dot{x}\Delta t + J_t^*(x(t), t)\Delta t + \text{higher order} \\
&= J^*(x(t), t) + \min_{u(t)}[J_x^*(x(t), t) f(x(t), u(t), t)\Delta t] + J_t^*(x(t), t)\Delta t + \text{higher order}
\end{aligned}$$

- If the first-order part of $J^*(x(t+\Delta t), t+\Delta t)$ is applied to the minimum cost, then

$$\begin{aligned} J^*(x(t),t) =& \min_{u(\tau)} \left[\int_t^{t+\Delta t} g(x(\tau),u(\tau),\tau)d\tau \right] \\ &+ J^*(x(t),t) + \min_{u(t)} [J_x^*(x(t),t) f(x(t),u(t),t)\Delta t] + J_t^*(x(t),t)\Delta t + \text{higher order} \\ &\Downarrow \\ 0 =& \min_{u(\tau)} \left[\int_t^{t+\Delta t} g(x(\tau),u(\tau),\tau)d\tau + J_x^*(x(t),t) f(x(t),u(t),t)\Delta t \right] + J_t^*(x(t),t)\Delta t + \text{higher order} \end{aligned}$$

- Dividing by Δt and taking the limit as $\Delta t \to 0$, we have

$$J_t^*(x(t),t) + \min_{u(t)} [g(x(t),u(t),t) + J_x^*(x(t),t) f(x(t),u(t),t)] = 0$$

- If the minimum cost is assigned to the Lyapunov function $J^*(x,t) = V(x,t)$, then we have the HJB equation as follows:

$$\therefore \quad V_t(x,t) + \min_u [g(x,u,t) + V_x(x,t) f(x,u,t)] = 0 \tag{83}$$

4. (Example 3.3) Obtain the HJB equation of the system with the performance index as follows:

$$\dot{x} = A(x,t)x + B(x,t)u \qquad J = \lim_{t\to\infty}\left[V(x(t),t) + \frac{1}{2}\int_0^t x^T Q(x,\tau)x + u^T R(x,\tau)u \, d\tau\right] \tag{84}$$

with $Q(x,t) = Q^T(x,t) \geq 0$ and $R(x,t) = R^T(x,t) > 0$ for all $t \geq 0$ and x.

- The Hamiltonian quantity is

$$H(x,u,V_x,t) = \frac{1}{2}x^T Q(x,t)x + \frac{1}{2}u^T R(x,t)u + V_x(x,t)[A(x,t)x + B(x,t)u]$$

- Its minimum is achieved

$$\frac{\partial H}{\partial u} = R(x,t)u + B^T(x,t)V_x^T(x,t) = 0 \qquad \Rightarrow \qquad u = -R^{-1}(x,t)B^T(x,t)V_x^T$$

- The HJB equation (82) is obtained as

$$V_t(x,t) + \min_u H(x,u,V_x,t) = 0$$

$$\Downarrow$$

$$\therefore \quad V_t(x,t) + V_x(x,t)A(x,t)x - \frac{1}{2}V_x(x,t)B(x,t)R^{-1}(x,t)B^T(x,t)V_x^T(x,t) + \frac{1}{2}x^T Q(x,t)x = 0$$

- After solving the HJB equation, we can determine the optimal control input, but it is not easy to solve because the it is nonlinear partial differential equation.
- The control input $u^* = -R^{-1}(x,t)B^T(x,t)V_x^T$ is called ***LQ control*** in the case of linear system because the performance index follows the linear quadratic form.

(PID) 3.4 HJI Equation

1. Hamilton-Jacobi-Isaacs (HJI) equation is similar with HJB equation!

2. For given system and performance index to be optimized:

$$\dot{x} = f(x, u, w, t) \qquad J = \lim_{t\to\infty}\left[V(x(t), t) + \int_0^t g(x(\tau), u(\tau), w(\tau), \tau)d\tau\right] \tag{85}$$

   if the Hamiltonian quantity is defined

$$H(x, u, w, V_x, t) = g(x, u, w, t) + V_x(x, t)f(x, u, w, t) \tag{86}$$

   then Hamilton-Jacobi-Isaacs (HJI) equation is obtained

$$\therefore \quad V_t + \min_u \max_w H(x, u, w, V_x, t) = 0 \qquad \Leftarrow \qquad \frac{\partial H}{\partial u} = 0 \left(\frac{\partial^2 H}{\partial u^2} > 0\right) \text{ and } \frac{\partial H}{\partial w} = 0 \left(\frac{\partial^2 H}{\partial w^2} < 0\right) \tag{87}$$

3. $u(t)$ is a player (control input) to minimize the cost J while $w(t)$ is an opponent (permissible disturbance input) to maximize the allowable cost.

4. (Example 3.4) Obtain the HJI equation of the system with the performance index as follows:

$$\dot{x} = A(x,t)x + B_1(x,t)u + B_2(x,t)w \qquad J = \lim_{t\to\infty}\left[V(x(t),t) + \frac{1}{2}\int_0^t x^TQ(x,\tau)x + u^TR(x,\tau)u - \gamma^2 w^Tw\, d\tau\right] \tag{88}$$

with $Q(x,t) = Q^T(x,t) \geq 0$ and $R(x,t) = R^T(x,t) > 0$ for all $t \geq 0$ and x.

- The Hamiltonian quantity is

$$H(x,u,w,V_x,t) = \frac{1}{2}x^TQ(x,t)x + \frac{1}{2}u^TR(x,t)u - \frac{\gamma^2}{2}w^Tw + V_x(x,t)[A(x,t)x + B_1(x,t)u + B_2(x,t)w]$$

- Its minimum player and opponent are obtained

$$\frac{\partial H}{\partial u} = R(x,t)u + B_1^T(x,t)V_x^T(x,t) = 0 \qquad u = -R^{-1}(x,t)B_1^T(x,t)V_x^T$$
$$\frac{\partial H}{\partial w} = -\gamma^2 w + B_2^T(x,t)V_x^T(x,t) = 0 \qquad w = \frac{1}{\gamma^2}B_2^T(x,t)V_x^T$$

- The HJI equation (87) is obtained as

$$V_t + \min_u \max_w H(x,u,w,V_x,t) = 0$$
$$\Downarrow$$
$$\therefore \quad V_t(x,t) + V_x(x,t)A(x,t)x - \frac{1}{2}V_x(x,t)B_1(x,t)R^{-1}(x,t)B_1^T(x,t)V_x^T(x,t)$$
$$+ \frac{1}{2\gamma^2}V_x(x,t)B_2(x,t)B_2^T(x,t)V_x^T(x,t) + \frac{1}{2}x^TQ(x,t)x = 0$$

- After solving the HJI equation, we can determine both the optimal control input and the allowable maximal disturbance input, but it is not easy to solve because the it is nonlinear partial differential equation.

- The control input $u^* = -R^{-1}(x,t)B_1^T(x,t)V_x^T$ is called $\mathcal{H}_\infty$ *control* in the case of linear system or $\mathcal{L}_2$ *gain control* in the case of nonlinear system because $\mathcal{L}_2$-gain from the disturbance input to the state is $\leq \gamma$.
- The disturbance input $w^* = \frac{1}{\gamma^2}B_2^T(x,t)V_x^T$ is called *worst-case disturbance* because it is a maximal allowable disturbance without affecting the stability.

5. (Example 3.5) For given scalar unstable system and the performance index,

$$\dot{x} = x + u + w \qquad\qquad J = \lim_{t\to\infty}\left[V(x(t)) + \frac{1}{2}\int_0^t x(\tau)^2 + u(\tau)^2 - \gamma^2 w(\tau)^2 \, d\tau\right],$$

(1) obtain LQ optimal controller by letting $w(t) = 0$?
(2) obtain $\mathcal{H}_\infty$ ($\mathcal{L}_2$-gain) optimal controller when $\gamma = \sqrt{2}$?
(3) compare the closed-loop responses of $\mathcal{H}_\infty$ optimal controller and LQ optimal controller against the disturbance input $w(t) = \delta(t)$ and zero initial condition $x(0) = 0$?
(4) show that $\mathcal{H}_\infty$ optimal controller becomes equal to LQ optimal controller as $\gamma \to \infty$

(Solution)

(1) In (Example 3.1), we already obtained the LQ optimal controller when $w(t) = 0$:

$$\therefore \quad u_{LQ} = -(1+\sqrt{2})x$$

(2) Hamiltonian function for a given system is

$$H(x,u,V_x) = \frac{1}{2}x^2 + \frac{1}{2}u^2 - \frac{\gamma^2}{2}w^2 + V_x(x+u+w)$$

To find the $\mathcal{H}_\infty$ optimal control input, let us differentiate the Hamiltonian function as follows:

$$\frac{\partial H}{\partial u} = u + V_x = 0 \quad \rightarrow \quad \therefore \quad u = -V_x$$

$$\frac{\partial^2 H}{\partial u^2} = 1 > 0 \quad \rightarrow \quad \mathcal{H}_\infty \text{ optimal control is achieved when } u^* = -V_x$$

Also, the worst-case disturbance can be found by using the following relation:

$$\frac{\partial H}{\partial w} = -\gamma^2 w + V_x = 0 \qquad \rightarrow \qquad \therefore \quad w = \frac{1}{\gamma^2} V_x$$

$$\frac{\partial^2 H}{\partial w^2} = -\gamma^2 < 0 \qquad \rightarrow \qquad \text{worst-case disturbance is achieved (occurred) when } \; w^* = \frac{1}{\gamma^2} V_x$$

Now the HJI equation is obtained as follow:

$$V_t + \min_u \max_w H(x, u, w, V_x, t) = 0$$

$$\downarrow$$

$$V_t + V_x x - \frac{1}{2} V_x^2 + \frac{1}{2\gamma^2} V_x^2 + \frac{1}{2} x^2 = 0$$

In order to solve the HJI equation, let us assume $V(x) = \frac{1}{2} p x^2$ with the positive constant $p > 0$. Then we have $V_t = 0$, $V_x = px$ and

$$V_t + V_x x - \frac{1}{2} V_x^2 + \frac{1}{2\gamma^2} V_x^2 + \frac{1}{2} x^2 = p x^2 - \frac{1}{2} p^2 x^2 + \frac{1}{2\gamma^2} p^2 x^2 + \frac{1}{2} x^2 = \left(p - \frac{1}{2} p^2 + \frac{1}{2\gamma^2} p^2 + \frac{1}{2} \right) x^2 = 0$$

Since $x \neq 0$, we can get the following

$$\left(1 - \frac{1}{\gamma^2} \right) p^2 - 2p - 1 = 0 \qquad \rightarrow \qquad p = \frac{\gamma}{\gamma^2 - 1} (\gamma \pm \sqrt{2\gamma^2 - 1})$$

Here, since $\gamma = \sqrt{2}$, the positive constant p can be determined as follow

$$p = 2 + \sqrt{6}$$

Finally, the $\mathcal{H}_\infty$ optimal controller is

$$\therefore \qquad u_\infty = -(2+\sqrt{6})x \quad \text{when } \gamma = \sqrt{2}$$

(3) The closed-loop equation can be obtained by applying the $\mathcal{H}_\infty$ optimal controller to the system as follows:

$$\dot{x} = x - (2+\sqrt{6})x + w = -(1+\sqrt{6})x + w$$

To obtain the response of above differential equation, we take the Laplace transform

$$sX(s) - x(0) = -(1+\sqrt{6})X(s) + W(s) \qquad \rightarrow \qquad X(s) = \frac{1}{s+(1+\sqrt{6})} \quad \text{since } W(s) = 1$$

Take an inverse Laplace transform to obtain the response, then

$$\therefore \qquad x_\infty(t) = e^{-(1+\sqrt{6})t} \quad \text{for } \; t \geq 0$$

On the other hand, the closed-loop equation can be obtained by applying the LQ optimal controller to the system as follows:

$$\dot{x} = x - (1+\sqrt{2})x + w = -\sqrt{2}x + w$$

To obtain the response of above differential equation, we take the Laplace transform

$$sX(s) - x(0) = -\sqrt{2}X(s) + W(s) \qquad \rightarrow \qquad X(s) = \frac{1}{s+\sqrt{2}} \quad \text{since } W(s) = 1$$

Take an inverse Laplace transform to obtain the response, then

$$\therefore \qquad x_{LQ}(t) = e^{-\sqrt{2}t} \quad \text{for } \; t \geq 0$$

As we can see in the closed-loop responses of $x_\infty(t)$ and $x_{LQ}(t)$, the $\mathcal{H}_\infty$ controller shows the better performance against the disturbances because $(1+\sqrt{6}) > \sqrt{2}$

(4) As $\gamma \to \infty$,

$$p = \lim_{\gamma\to\infty}\left[\frac{\gamma^2}{\gamma^2-1} + \frac{\gamma\sqrt{2\gamma^2-1}}{\gamma^2-1}\right] \approx 1+\sqrt{2}$$

Hence, the $\mathcal{H}_\infty$ controller becomes equal to the form of LQ optimal controller as $\gamma \to \infty$:

$$\therefore \quad u_\infty \approx u_{LQ} = -(1+\sqrt{2})x \quad \text{as } \gamma \to \infty$$

(PID) HJB and HJI Equations

1. HJB equation

   - Generalized Hamilton-Jacobi theory [1967]
   - Multi-variable system and combinational problem
   - Dynamic programming
   - First order nonlinear PDE
   - (in the case of linear system) Riccati equation for LQ control problem

2. HJI equation

   - Isaacs theory [1975]
   - Two player differential game theory
   - Minimizing player : control input (Opponent : disturbance input)
   - (in the case of linear system) Riccati equation for $\mathcal{H}_\infty$ control problem

- (HW # 7) solve 4 problems 3.2, 3.3, 3.6, and 3.8

(PID) 4 Regulation and Tracking Control

1. There are two types of controllers according to the way that target is given.
2. Some targets might be given as a constant force or position, but some targets should be specified as a function of time.
3. We classify the controllers;
   - *set-point regulation control* when the constant target is given,
   - *trajectory tracking control* when the target profile is given as function of time.
4. The set-point regulation control belongs to the *time-invariant* control system, but the trajectory tracking control to the *time-varying* control system.

(PID) 4.1 Global Asymptotic Stability of Set-Point PD Control

1. Let us consider the set-point regulation (point-to-point) control for Hamiltonian systems.

2. For given Hamiltonian system (62), let us assume that there is no gravity force $g(q) = 0$

$$\dot{q} = M^{-1}(q)p \qquad \dot{p} = C^T(q,\dot{q})M^{-1}(q)p + \tau.$$

3. Putting a PD control as following form:

$$\tau = -K_D\dot{q} - K_P(q - q_s) \tag{89}$$

into the Hamiltonian equation of motion, the closed-loop control system is obtained as follows:

$$\dot{q} = M^{-1}(q)p \qquad \dot{p} = C^T(q,\dot{q})M^{-1}(q)p - K_D M^{-1}(q)p - K_P(q - q_s). \tag{90}$$

where $K_D > 0, K_P > 0$ are constant diagonal gain matrices, q_s is the set-point (target position) to be controlled, and $p = 0, q = q_s$ is the equilibrium point.

4. Consider Lyapunov function composed of kinetic energy and potential energy by spring effect due to K_P term

$$V(p,q) = \frac{1}{2}p^T M^{-1}(q)p + \frac{1}{2}(q - q_s)^T K_P(q - q_s) \tag{91}$$

then its time derivative is obtained along the control system trajectory of (90) as follows:

$$\begin{aligned}\dot{V}(p,q) &= p^T M^{-1}(q)\dot{p} + \frac{1}{2}p^T \dot{M}^{-1}(q)p + (q - q_s)^T K_P\dot{q} \\ &= p^T M^{-1}(q)C^T(q,\dot{q})M^{-1}(q)p - p^T M^{-1}(q)K_D M^{-1}(q)p + \frac{1}{2}p^T \dot{M}^{-1}(q)p.\end{aligned}$$

Since we know that $\dot{M}^{-1} = -M^{-1}\dot{M}M^{-1}$ and $\dot{M}(q) - 2C^T(q,\dot{q})$ is skew symmetric, it is easy to see that

$$\dot{V}(p,q) = \frac{1}{2}p^T M^{-1}(q)\left[2C^T(q,\dot{q}) - \dot{M}(q)\right]M^{-1}(q)p - p^T M^{-1}(q)K_D M^{-1}(q)p$$
$$= -p^T M^{-1}(q)K_D M^{-1}(q)p = \begin{bmatrix} p^T & q^T \end{bmatrix}\begin{bmatrix} -M^{-1}(q)K_D M^{-1}(q) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix} \leq 0 \qquad \textbf{(92)}$$

5. If there exists such a function $V(x)$ defined in a certain domain Ω of the state space of $x = (p,q)$ containing the equilibrium point $x_0 = (0, q_s)$, then for any initial condition $x(0) = (p(0), q(0))$ in a neighborhood of x_0, the control system trajectory $(p(t), q(t))$ of equation (90) approaches asymptotically to the *maximal invariant set* M contained in the set

$$E = \left\{x = (p,q) \in \Omega | \dot{V} = 0\right\}.$$

In our case, according to equation (92), $\dot{V} = 0$ means $p = 0$ because $K_D > 0$. Therefore, it holds along any control system trajectory in E that

$$\dot{p} = -K_P(q - q_s).$$

This in turn implies that M is composed of the single point $x_0 = (p = 0, q = q_s)$. Also, since Lyapunov function (91) is unbounded function for x, *i.e.*, $V \to \infty$ as $x \to \infty$, the *global asymptotic stability* (GAS) of the equilibrium point $x_0 = (0, q_s)$ can be proved by PD control.

(PID) 4.2 $\mathcal{H}_\infty$ Control for Trajectory Tracking

1. Trajectory tracking control is basically different from the set-point regulation control in that the desired configuration $q_d(t)$, velocity $\dot{q}_d(t)$ and its acceleration $\ddot{q}_d(t)$ profiles as time-varying functions are added to typical Lagrangian systems.

2. For given Lagrangian system (57) of $M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$, let us define the errors as state vectors:

$$e \triangleq q_d - q \qquad\qquad \dot{e} \triangleq \dot{q}_d - \dot{q}$$

   and then, if we adopt the computed-torque control (CTC) as following form:

$$\tau = \widehat{M}(q)(\ddot{q}_d + K_P\dot{e} + K_I e) + \widehat{C}(q,\dot{q})(\dot{q}_d + K_P e + K_I \int e dt) + \widehat{g}(q) - u, \tag{93}$$

   where u is the *auxiliary control input* to be designed later and $(\widehat{M}, \widehat{C}, \widehat{g})$ are the *estimates* for the actual (M, C, g)

3. The resultant system dynamics is given by

$$M(q)(\ddot{e}+K_P\dot{e}+K_Ie)+C(q,\dot{q})(\dot{e}+K_Pe+K_I\int edt)=u+w, \tag{94}$$

with the disturbance vector w including the model uncertainties as follows:

$$w=[M(q)-\widehat{M}(q)](\ddot{q}_d+K_P\dot{e}+K_Ie)+[C(q,\dot{q})-\widehat{C}(q,\dot{q})](\dot{q}_d+K_Pe+K_I\int edt)+[g(q)-\widehat{g}(q)]. \tag{95}$$

4. For given system (94), if we define the state vector like this

$$x \triangleq \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \int edt \\ e \\ \dot{e} \end{bmatrix} \in \Re^{3n},$$

then the state space representation of the system (94) can be simply written by

$$\dot{x}=A(x,t)x+B(x,t)w+B(x,t)u \tag{96}$$

where

$$A(x,t)=\begin{bmatrix} 0_{n\times n} & I_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & I_{n\times n} \\ -M^{-1}CK_I & -M^{-1}CK_P-K_I & -M^{-1}C-K_P \end{bmatrix} \tag{97}$$

$$B(x,t)=\begin{bmatrix} 0_{n\times n} \\ 0_{n\times n} \\ M^{-1} \end{bmatrix}, \tag{98}$$

in which $0_{n\times n}$ and $I_{n\times n}$ are $n\times n$ zero and identity matrices, respectively. From now on, we will omit the dimension of the matrix if its dimension can be estimated.

5. (Lemma 4.1) (HJI) For given system model (96), suppose there exists a smooth function $V(x,t) > 0$ with $V(0,t) = 0$ that satisfies

$$HJI = V_t + V_x Ax + \frac{1}{2\gamma^2} V_x BB^T V_x^T - \frac{1}{2} V_x BR^{-1}B^T V_x^T + \frac{1}{2} x^T Qx = 0, \tag{99}$$

where $V_t = \frac{\partial V}{\partial t}$, $V_x = \frac{\partial V}{\partial x^T}$ and $\gamma > 0$, then the auxiliary control

$$u = -R^{-1}B^T V_x^T \tag{100}$$

minimizes the performance index as following form:

$$J = \lim_{t\to\infty} \left[V(x,t) + \frac{1}{2} \int_0^t \left[x^T Qx + u^T Ru - \gamma^2 w^T w \right] dt \right] \tag{101}$$

6. (Proof) Let us consider the Lyapunov function:

$$V(x,t) = \frac{1}{2} x^T P(x,t) x$$

with the concrete form of $P(x,t)$ matrix as following form:

$$P(x,t) = \begin{bmatrix} K_I M(q) K_I + K_I K_P K & K_I M(q) K_P + K_I K & K_I M \\ K_P M(q) K_I + K_I K & K_P M(q) K_P + K_P K & K_P M(q) \\ M(q) K_I & M(q) K_P & M(q) \end{bmatrix}, \tag{102}$$

notice that $M(q) = M(q_d(t) - x_2)$ is function of time and x_2, where the positive definiteness of $P(x,t)$ requires the following conditions

a) $K > 0, K_P > 0, K_I > 0$ are constant diagonal matrices,

b) $K_P^2 > 2K_I$.

From now on, we would like to obtain the differential Riccati equation from HJI (99).

- Note that

$$V_t = \frac{1}{2}x^T \frac{\partial P}{\partial t} x, \qquad V_x = \frac{1}{2}x^T \frac{\partial P}{\partial x^T} x + x^T P, \tag{103}$$

- Fortunately, since $P(x,t)$ is not a function of $x_1 = \int e dt$ and $x_3 = \dot{e}$, the V_x can be simplified as:

$$V_x = \frac{1}{2}x^T \begin{bmatrix} 0 & \frac{\partial P}{\partial x_2^T}x & 0 \end{bmatrix} + x^T P.$$

- Now we have

$$V_x A x = \frac{1}{2}x^T P A x + \frac{1}{2}x^T A^T P x + \frac{1}{2}x^T \begin{bmatrix} 0 & \frac{\partial P}{\partial x_2^T}x & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ (*) \end{bmatrix} = \frac{1}{2}x^T \left\{ PA + A^T P + \frac{\partial P}{\partial x_2^T}\dot{x}_2 \right\} x$$

$$V_t + V_x A x = \frac{1}{2}x^T \left\{ \frac{\partial P}{\partial t} + \frac{\partial P}{\partial x_2^T}\dot{x}_2 + PA + A^T P \right\} x = \frac{1}{2}x^T \left\{ \dot{P} + PA + A^T P \right\} x$$

$$V_x B = x^T P B + \frac{1}{2}x^T \left[\frac{\partial P}{\partial x^T} x \right] B = x^T P B, \quad \text{because} \quad \left[\frac{\partial P}{\partial x^T} x \right] B = \begin{bmatrix} 0 & \frac{\partial P}{\partial x_2^T}x & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ M^{-1} \end{bmatrix} = 0.$$

- Hence the HJI equation (99) can be rewritten as:

$$\frac{1}{2}x^T \left\{ \dot{P} + PA + A^T P - PBR^{-1}B^T P + \frac{1}{\gamma^2} PBB^T P + Q \right\} x = 0,$$

- For any $x \neq 0$, the HJI equation is converted into the differential Riccati equation

$$\dot{P} + PA + A^T P - PBR^{-1}B^T P + \frac{1}{\gamma^2} PBB^T P + Q = 0 \tag{104}$$

- For arbitrarily given weighting matrices Q and R, the Lyapunov matrix (102) does not always satisfy the differential Riccati equation (104) as well. The weighting matrices Q and R will be inversely found from the differential Riccati equation.

7. (Theorem 4.1) Assume that there exists a Lyapunov matrix $P(x,t)$ (102) for Lagrangian systems (96). If the *control input weighting* is defined as following matrix:

$$R = \left(K + \frac{1}{\gamma^2}I\right)^{-1}, \tag{105}$$

then the *state weighting* matrix can be inversely obtained from the differential Riccati equation (104) as follows:

$$Q = \begin{bmatrix} K_I^2 K & 0 & 0 \\ 0 & (K_P^2 - 2K_I)K & 0 \\ 0 & 0 & K \end{bmatrix} > 0, \tag{106}$$

where Q is a positive definite, diagonal and constant matrix.

8. (Proof) By using the definition of R of Eq. (105), the differential Riccati equation (104) can be simplified to

$$\dot{P} + PA + A^T P - PB\left(R^{-1} - \frac{1}{\gamma^2}I\right)B^T P + Q = 0$$
$$\Downarrow$$
$$\dot{P} + A^T P + PA - PBKB^T P + Q = 0 \tag{107}$$

By using the characteristics $\dot{M} - C^T - C = 0$ of Lagrangian system, the following equation can be firstly computed

$$\dot{P} + A^T P + PA = \begin{bmatrix} 0 & K_I K_P K & K_I K \\ K_I K_P K & 2K_I K & K_P K \\ K_I K & K_P K & 0 \end{bmatrix}.$$

Also, the remaining term is

$$PBKB^TP = \begin{bmatrix} K_I^2K & K_IK_PK & K_IK \\ K_IK_PK & K_P^2K & K_PK \\ K_IK & K_PK & K \end{bmatrix}.$$

Hence, the matrix Q found from (107) has the form of (106).

$$\begin{aligned} Q &= -(\dot{P} + A^TP + PA) + PBKB^TP \\ &= \begin{bmatrix} 0 & -K_IK_PK & -K_IK \\ -K_IK_PK & -2K_IK & -K_PK \\ -K_IK & -K_PK & 0 \end{bmatrix} + \begin{bmatrix} K_I^2K & K_IK_PK & K_IK \\ K_IK_PK & K_P^2K & K_PK \\ K_IK & K_PK & K \end{bmatrix} \\ &= \begin{bmatrix} K_I^2K & 0 & 0 \\ 0 & (K_P^2 - 2K_I)K & 0 \\ 0 & 0 & K \end{bmatrix} > 0 \end{aligned}$$

where since $K_P^2 > 2K_I$, it is a positive definite, diagonal and constant matrix.

9. For the constant weighting matrices Q and R given in above Theorem, since the Lyapunov matrix (102) satisfies the differential Riccati equation (104), the auxiliary control input (100) is a $\mathcal{H}_\infty$ controller for the given performance index (101), also, the computed-torque controller of (93) has the following form:

$$\begin{aligned}\tau =& \widehat{M}(q)(\ddot{q}_d + K_P\dot{e} + K_I e) + \widehat{C}(q,\dot{q})(\dot{q}_d + K_P e + K_I \int e dt) + \widehat{g}(q) \\ &+ \left(K + \frac{1}{\gamma^2} I\right)\left(\dot{e} + K_P e + K_I \int e dt\right),\end{aligned} \tag{108}$$

where we should notice that the auxiliary controller (100) has the form of PID one.

- (HW # 8) solve example 4.1, example 4.2, and prove the following two equations

$$\dot{P} + A^T P + PA = \begin{bmatrix} 0 & K_I K_P K & K_I K \\ K_I K_P K & 2K_I K & K_P K \\ K_I K & K_P K & 0 \end{bmatrix}.$$

and

$$PBKB^T P = \begin{bmatrix} K_I^2 K & K_I K_P K & K_I K \\ K_I K_P K & K_P^2 K & K_P K \\ K_I K & K_P K & K \end{bmatrix}.$$