3 Design using Discrete Equivalents

- It is important to remember that how to convert $D_c(s)$ into $D_d(z)$ is approximation; there is no exact solution for all possible inputs because $D_c(s)$ responds to the complete time history of $e(t)$, whereas $D_d(z)$ has access to only the samples $e(kT)$.

- (8.3.1) Tustin's Method

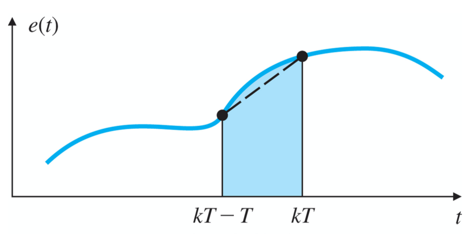
  1. Tustin's method is a digitization technique that approaches the problem as one of numerical integration. Suppose

     $$\frac{U(s)}{E(s)} = D_c(s) = \frac{1}{s}$$

     which is integration. Therefore, it is corresponding to the *trapezoidal integration* as follows:

     $$\begin{aligned} u(kT) &= \int_0^{kT-T} e(t)dt + \int_{kT-T}^{kT} e(t)dt \\ &= u(kT-T) + \text{area under } e(t) \text{ over last period, } T, \\ u(k) &= u(k-1) + T\frac{[e(k-1)+e(k)]}{2} \end{aligned}$$

     where T is the sample period.



2. Taking z-transform,

$$\frac{U(z)}{E(z)} = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}} = \frac{1}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

3. In fact, the Tustin's method approximates $z = e^{sT}$ as follows:

$$s \approx \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

where it can be derived from the Taylor's series expansions as follows:

$$z = e^{sT} = \frac{e^{\frac{sT}{2}}}{e^{-\frac{sT}{2}}} = \frac{1+\frac{sT}{2}+\frac{s^2T^2}{2^2}+\cdots}{1-\frac{sT}{2}+\frac{s^2T^2}{2^2}-\cdots} \approx \frac{1+\frac{sT}{2}}{1-\frac{sT}{2}} = \frac{2+sT}{2-sT} \qquad \rightarrow \qquad s \approx \frac{2}{T}\frac{z-1}{z+1}$$

4. For $D_c(s) = \frac{a}{s+a}$ as an example, we have

$$D_d(z) = \frac{U(z)}{E(z)} = \frac{a}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}+a} = \frac{aT(1+z^{-1})}{2(1-z^{-1})+aT(1+z^{-1})} = \frac{aT(1+z^{-1})}{(2+aT)-(2-aT)z^{-1}}$$

$$(2+aT)u(k) - (2-aT)u(k-1) = aT[e(k)+e(k-1)]$$

$$u(k) = \frac{(2-aT)}{(2+aT)}u(k-1) + \frac{aT}{(2+aT)}[e(k)+e(k-1)]$$

5. (Example 8.1) Determine the difference equation with a sample rate of 25 times bandwidth using Tustin's approximation.

$$D_c(s) = 10\frac{s/2+1}{s/10+1}$$

Since the bandwidth is approximately $\omega_{bd} = 10[rad/s]$, the sampling rate should be

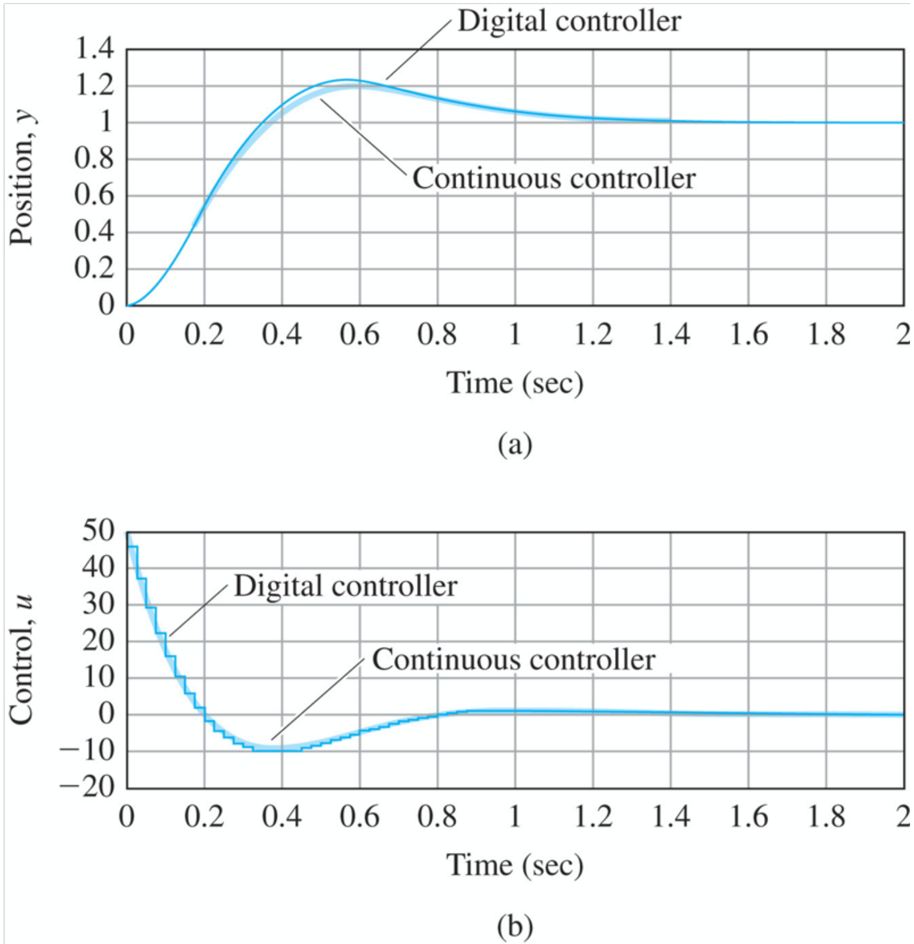
$$\omega_s = 25 \times \omega_{bd} = 250[rad/s] \qquad \rightarrow \qquad f_s = \frac{\omega_s}{2\pi} \approx 40[Hz] \qquad \rightarrow \qquad T = \frac{1}{f_s} = \frac{1}{40} = 0.025[s]$$

The difference TF can be obtained as

$$\begin{aligned} D_d(z) &= 10\frac{\frac{1}{T}\frac{1-z^{-1}}{1+z^{-1}}+1}{\frac{1}{5T}\frac{1-z^{-1}}{1+z^{-1}}+1} = 10\frac{5(1-z^{-1})+5T(1+z^{-1})}{(1-z^{-1})+5T(1+z^{-1})} \\ &= 50\frac{(1+T)-(1-T)z^{-1}}{(1+5T)-(1-5T)z^{-1}} = 50\frac{1.025-0.975z^{-1}}{1.125-0.875z^{-1}} = \frac{45.556-43.333z^{-1}}{1-0.778z^{-1}} \end{aligned}$$

Finally, the difference equation is

$$u(k) = 0.778u(k-1) + 45.556[e(k) - 0.951e(k-1)]$$


Digital controller
Continuous controller
Position, y
Time (sec)
(a)
Digital controller
Continuous controller
Control, u
Time (sec)
(b)

- (8.3.2) Zeroth-Order Hold (ZOH) Method

  1. Tustin's method essentially assumed that the input to the controller varied linearly early between the past sample and the current sample.
  2. Another assumption is that the input to the controller remains constant throughout the sample period. $\rightarrow$ ZOH
  3. One input sample produces a square pulse of height $e(k)$ that lasts for one sample period T.
  4. For a constant positive step input, $e(k)$, at time k, $E(s) = e(k)/s$, so the result would be

     $$D_d(z) = \mathcal{Z}\left(\frac{D_c(s)}{s}\right)$$

     Furthermore, a constant negative step, one cycle delayed, would be

     $$D_d(z) = z^{-1}\mathcal{Z}\left(\frac{D_c(s)}{s}\right)$$

     Therefore, the discrete TF for the square pulse is

     $$D_d(z) = (1 - z^{-1})\mathcal{Z}\left(\frac{D_c(s)}{s}\right)$$

5. (Example 8.2) Determine the difference equation with a sample period $T = 0.025[s]$ using ZOH approximation.

$$D_c(s) = 10\frac{s/2+1}{s/10+1} = 10\frac{5s+10}{s+10}$$

The discrete TF using ZOH with $aT = 0.25$ and $e^{-aT} = 0.779$ is

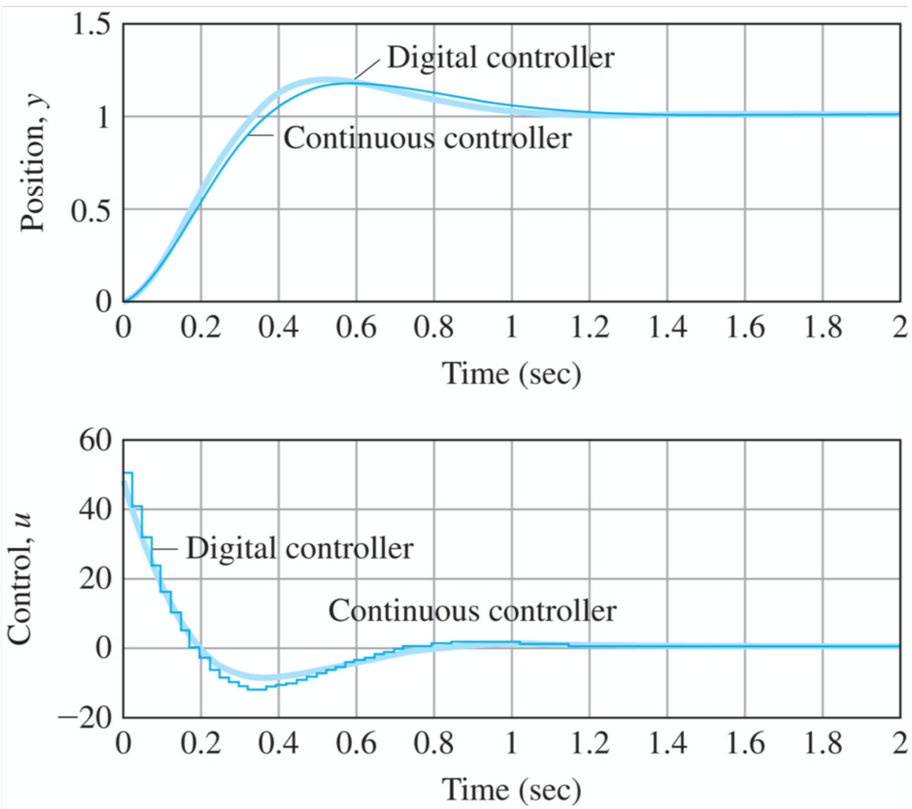
$$\begin{aligned}
D_d(z) = 10(1-z^{-1})\mathcal{Z}\left(\frac{5s+10}{s(s+10)}\right) &= 10(1-z^{-1})\mathcal{Z}\left(\frac{5}{s+10} + \frac{10}{s(s+10)}\right) \\
&= 10(1-z^{-1})\left(\frac{5}{1-e^{-0.25}z^{-1}} + \frac{z^{-1}(1-e^{-0.25})}{(1-z^{-1})(1-e^{-0.25}z^{-1})}\right) \\
&= 10(1-z^{-1})\left(\frac{5(1-z^{-1}) + z^{-1}(1-e^{-0.25})}{(1-z^{-1})(1-e^{-0.25}z^{-1})}\right) \\
&= \frac{50-47.79z^{-1}}{1-0.779z^{-1}}
\end{aligned}$$

Or,

$$\begin{aligned}
D_d(z) = 10(1-z^{-1})\mathcal{Z}\left(\frac{5s+10}{s(s+10)}\right) &= 10(1-z^{-1})\mathcal{Z}\left(\frac{1}{s} + \frac{4}{s+10}\right) \\
&= 10(1-z^{-1})\left(\frac{1}{1-z^{-1}} + \frac{4}{1-e^{-0.25}z^{-1}}\right) \\
&= 10(1-z^{-1})\left(\frac{(1-e^{-0.25}z^{-1}) + 4(1-z^{-1})}{(1-z^{-1})(1-e^{-0.25}z^{-1})}\right) \\
&= \frac{50-47.79z^{-1}}{1-0.779z^{-1}}
\end{aligned}$$

Finally, the difference equation is

$$\begin{aligned} u(k) &= 0.779u(k-1) + 50e(k) - 47.79e(k-1) \\ &= 0.779u(k-1) + 50[e(k) - 0.956e(k-1)] \end{aligned}$$



- (8.3.3) Matched Pole-Zero (MPZ) Method

  1. Another digitization method, called the matched pole-zero (MPZ) method, is suggested by matching the poles and zeros between s and z planes, using $z = e^{sT}$.
  2. Because physical systems often have more poles than zeros, it is useful to arbitrarily add zeros at $z = -1$, resulting in a $(1 + z^{-1})$ term in $D_d(z)$.
     a) Map poles and zeros according to the relation $z = e^{sT}$
     b) If the numerator is of lower order than the denominator, add powers of $(1 + z^{-1})$ to the numerator until numerator and denominator are of equal order.
     c) Set the DC or low frequency gain of $D_d(z)$ equal to that of $D_c(s)$.
  3. For example, the MPZ approximation

$$D_c(s) = K_c \frac{s+a}{s+b} \qquad D_d(z) = K_d \frac{1 - e^{-aT} z^{-1}}{1 - e^{-bT} z^{-1}}$$

where K_d is found by the DC-gain

$$\lim_{s \to 0} D_c(s) = K_c \frac{a}{b} \qquad \rightleftarrows \qquad \lim_{z \to 1} D_d(z) = K_d \frac{1 - e^{-aT}}{1 - e^{-bT}}$$

Thus the result is

$$K_d = K_c \frac{a}{b} \left(\frac{1 - e^{-bT}}{1 - e^{-aT}} \right)$$

4. As another example, the MPZ approximation

$$D_c(s) = K_c \frac{s+a}{s(s+b)} \qquad D_d(z) = K_d \frac{(1+z^{-1})(1-e^{-aT}z^{-1})}{(1-z^{-1})(1-e^{-bT}z^{-1})}$$

where K_d is found by the DC-gain *by deleting the pure integration term* both sides

$$\lim_{s\to 0} sD_c(s) = K_c \frac{a}{b} \qquad \rightleftarrows \qquad \lim_{z\to 1}(z-1)D_d(z) = K_d \frac{2(1-e^{-aT})}{1-e^{-bT}}$$

The result is

$$K_d = K_c \frac{a}{2b}\left(\frac{1-e^{-bT}}{1-e^{-aT}}\right)$$

5. (Example 8.3) Design a digital controller to have a closed-loop natural frequency $\omega_n = 0.3$ and a damping ratio $\zeta = 0.7$ using MPZ digitization

$$G(s) = \frac{1}{s^2}$$

Let us assume that the lead compensator is used

$$D_c(s) = K_c \frac{s+b}{s+a}$$

Then, we have the characteristic equation

$$1 + G(s)D_c(s) = 1 + K_c \frac{s+b}{s^2(s+a)} = s^3 + as^2 + K_c s + K_c b$$

$$\alpha_c(s) = (s^2 + 0.42s + 0.09)(s + 1.58) = s^3 + 2s^2 + 0.7536s + 0.1422$$

with $a = 2$, $b = 0.19 \approx 0.2$, and $K_c = 0.7536 \approx 0.81$. Now we have the lead compensator:

$$D_c(s) = 0.81 \frac{s + 0.2}{s + 2}$$

Let us determine the sampling rate and sampling period as follows:

$$\omega_s = 0.3 \times 20 = 6[rad/s] \quad \rightarrow \quad f_s = \frac{\omega_s}{2\pi} \approx 1[Hz] \quad \rightarrow \quad T = 1[s]$$

The MPZ digitization yields

$$D_d(z) = K_d \frac{1 - e^{-0.2} z^{-1}}{1 - e^{-2} z^{-1}} = K_d \frac{1 - 0.818 z^{-1}}{1 - 0.135 z^{-1}}$$

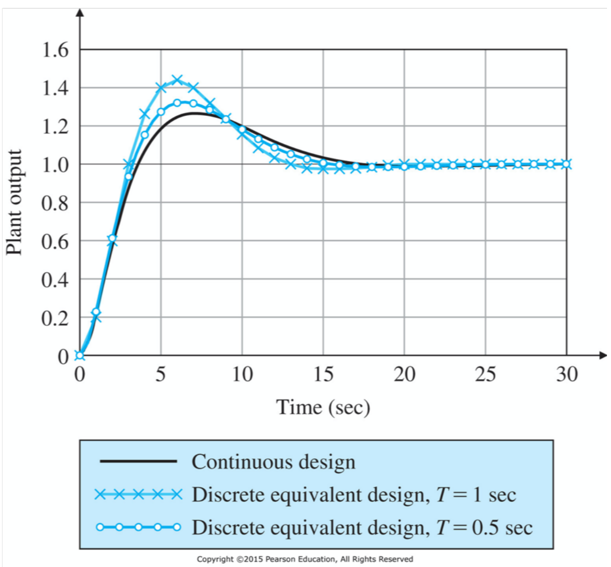
where the final value theorem gives

$$0.81 \frac{0.2}{2} = K_d \frac{1 - 0.818}{1 - 0.135} \qquad \rightarrow \qquad K_d = 0.385$$

The difference equation becomes

$$u(k) = 0.135 u(k-1) + 0.385[e(k) - 0.818 e(k-1)]$$

For the step responses,



- (8.3.4) Modified Matched Pole-Zero (MMPZ) Method

  1. Modify Step 2 in the MPZ so that the numerator is of lower order than denominator by 1. For example, if

$$D_c(s) = K_c \frac{s+a}{s(s+b)}$$

we skip Step 2 to get

$$D_d(z) = K_d \frac{z^{-1}(1-e^{-aT}z^{-1})}{(1-z^{-1})(1-e^{-bT}z^{-1})} \qquad \text{where} \quad K_d = K_c \frac{a}{b}\left(\frac{1-e^{-bT}}{1-e^{-aT}}\right)$$

We can see the difference equation as follow:

$$u(k) = (1+e^{-bT})u(k-1) - e^{-bT}u(k-2) + K_d[e(k-1) - e^{-aT}e(k-2)]$$

where it makes use of $e(k-1)$ that are one cycle old, not $e(k)$.

- (8.3.5) Comparison of Digital Approximation Methods

1. Let us compare four approximation methods with the sampling rate

$$D_c(s) = \frac{5}{s+5}$$

2. Tustin's method

$$\begin{aligned} D_d(z) &= \frac{5}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}+5} = \frac{5T(1+z^{-1})}{2(1-z^{-1})+5T(1+z^{-1})} = \frac{5T+5Tz^{-1}}{(2+5T)-(2-5T)z^{-1}} \\ &= \left(\frac{5T}{2+5T}\right)\frac{1+z^{-1}}{1-\left(\frac{2-5T}{2+5T}\right)z^{-1}} \end{aligned}$$

3. ZOH

$$\begin{aligned} D_d(z) &= (1-z^{-1})\mathcal{Z}\left(\frac{D_c(s)}{s}\right) = (1-z^{-1})\mathcal{Z}\left(\frac{5}{s(s+5)}\right) = (1-z^{-1})\frac{(1-e^{-5T})z^{-1}}{(1-z^{-1})(1-e^{-5T}z^{-1})} \\ &= (1-e^{-5T})\frac{z^{-1}}{1-e^{-5T}z^{-1}} \end{aligned}$$

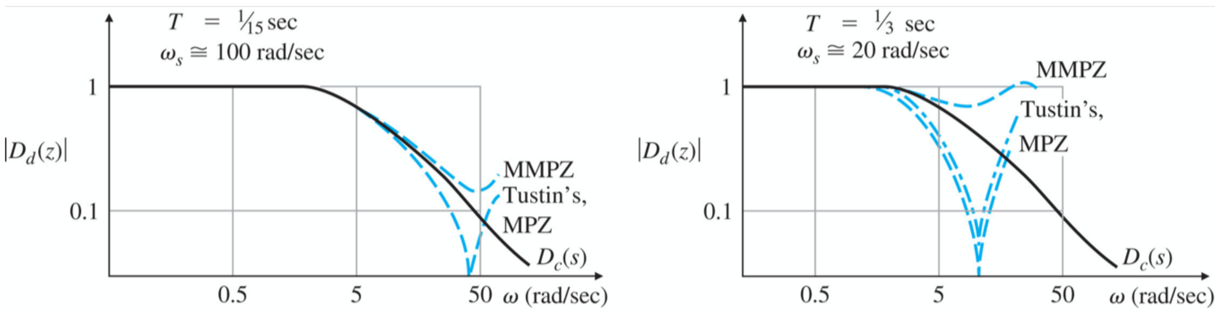
4. MPZ

$$\begin{aligned} D_d(z) &= K_d\frac{(1+z^{-1})}{1-e^{-5T}z^{-1}} \quad \textbf{where} \quad K_d\frac{2}{1-e^{-5T}} = 1 \\ &= \left(\frac{1-e^{-5T}}{2}\right)\frac{1+z^{-1}}{1-e^{-5T}z^{-1}} \end{aligned}$$

5. MMPZ

$$D_d(z) = K_d \frac{z^{-1}}{1 - e^{-5T} z^{-1}} \quad \text{where} \quad K_d \frac{1}{1 - e^{-5T}} = 1$$
$$= (1 - e^{-5T}) \frac{z^{-1}}{1 - e^{-5T} z^{-1}}$$

6. It is noted that Tustin and MPZ bring the similar structures each other, while ZOH and MMPZ show the similar structures, as shown in Table 8.2
7. Tustin and MPZ methods show a notch at $\omega_s/2$ because of their zero at $z = -1$ from $1 + z^{-1}$ term.



- (8.3.6) Applicability Limits of the Discrete Equivalent Design Method

1. The system can often be *unstable* for rates slower than approximately $5\omega_{bd}$, and
2. the damping would be *degraded* significantly for rates slower than about $10\omega_{bd}$
3. At sample rates $\geq 20\omega_{bd}$, design by discrete equivalent yields *reasonable* results, and
4. at sample rates of 25 times the bandwidth or higher, discrete equivalents can be used *with confidence*.
5. ZOH brings $T/2$ delay in the control system. A method to account for the $T/2$ delay is to include an approximation of the delay into the original plant model:

$$G_{ZOH}(s) = \frac{2/T}{s + 2/T}$$